Master of Business Administration

(Open and Distance Learning Mode)

Semester – II

Operation Research

Centre for Distance and Online Education (CDOE)

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OPERATIONS RESEARCH

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UNIT 1 : INTRODUCTION TO OPERATIONS RESEARCH

STRUCTURE

1.0 Objectives

1.1 Introduction

1.2 Concept of Operation Research (OR)

1.3 Scope of OR

1.4 Phases of OR

1.5 Application of OR

1.6 Limitations of OR

1.7 Summary

1.8 Key words

1.9 Self Assessment Questions

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1.0 OBJECTIVES

After studying this unit you should be able to:

* Explain the concept of OR;
* Asses the scope and phases of OR;
* Identify the applications of OR and
* Distinguish between models of OR.

1.1 INTRODUCTION

Operation research is the research done on operations. It can be visualized as a method, tool, set of technique, an activity that aids the manager in making a decision. It makes use of quantitative techniques to provide better solution to the problem.

The present day business scenario has vastly changed owing to the competition and increased complexity from all fronts. Globalization has made the domestic businesses also to move towards globally set benchmarks. Hence decision making has become a very complex and dynamic activity. It has to consider the various alternatives both qualitatively and quantitatively. The various techniques that are used to facilitate decision making are called quantitative methods or optimization techniques or decision science or operation analysis or operation research.

The concept of operation research was basically evolved during second world war in England. The major concern at that time was the efficient or the optimum use of scarce resources of war material including human resource. As this technique was mainly evolved for the military operations, it is known as operations research. With the conclusion of World war the application was OR was later spread to business to facilitate optimum use of resources such men, machine, money and material.

Meaning and Definitions : -

Operations research makes an attempt to solve many of business problems in quantitative way. Many authors have defined operations research in their own way.

You can go through the following definitions of operations research.

Operation Research – Meaning and Definitions

(*a*) Pocock stresses that OR is an applied science; he states "OR is scientific methodology-analytic-cal, experimental, quantitative which by assessing the overall implication of various alternative courses of action in a management system, provides (*b*) Morse and Kimball have stressed the quantitative approach of OR and have described it as "a scientific method of providing executive departments with a quantitative basis for decisions regarding the operations under their control".

(*c*) Miller and Starr see OR as applied decision theory. They state "OR is applied decision theory. It uses any scientific, mathematical or logical means to attempt to cope with the problems that confront the executive, when he tries to achieve a thorough—going rationality in dealing with his decision problem".

(*d*) Saaty considers OR as tool of improving the quality of answers to problems. He say, "OR is the art of giving bad answers to problems which otherwise have worse answers".

Few other definitions of OR are as follows:

- "OR is concerned with scientifically deciding how to best design and operate man-machine system usually requiring the allocation of scare resources."

  – Operations Research Society, America

- "OR is essentially a collection of mathematical techniques and tools which in conjunction with system approach, are applied to solve practical decision problems of an economic or engineering nature".

  – Daellenbach and George

- "OR utilizes the planned approach (updated scientific method) and an interdisciplinary team in order to represent complex functional relationships as mathematical models for the purpose of providing a quantitative analysis".

  – Thieraub and Klekamp

- "OR is a scientific knowledge through interdisciplinary team effort for the purpose of deter-mining the best utilization of limited resources."

  – H.A. Taha

- "OR is a scientific approach to problem solving for executive management".

  – H.M. Wagner

By going through the above definitions, we can interpret that Operation Research is scientific decision making tool leading to optimal use of resources. Operations research helps in taking business decisions more effectively and objectively. The game theory for example considers competition. The transportation problem tries identify the least cost route for transporting items from different sources to different destinations. The assignment problems helps in assigning the work to workers in such a way that the total time required complete the task would be minimized. The operation also helps in taking an objective decision which would either maximize the profit or minimize the cost by choosing a good alternative among the various alternative available.

1.2 CONCEPT OF OPERATIONS RESEARCH

The operations research attempts to give mathematics based solutions to the various problems that are encountered in the day to day business. The concept of operations research dates back to 19th century. In fact many of the operations research topics being discussed today were evolved much before the concept of operations research was evolved. The term Operations Research was coined during the Second World War. The Second World War started in 1939 soon after the First World War (1914-1918) faced a severe crunch of resources such as men, war materials, money. Hence they were supposed to have optimum utilization of the existing resources. Many researches were taken place in this area to find ways to gain most output with least input and hence obtain optimal use of resources. With the conclusion of Second World War, the operations research techniques found its own place in the business field where managers are interested to increase their productivity with minimum available resources.

1.3 SCOPE OF OPERATIONS RESEARCH

Operations research has applications in various fields. The scope of operations research is not only confirmed to business but also to medical, research, military, computers and so on.

i. **In Defence Operations :**

   In modern warfare, the defense operations are carried out by three major independent components namely Air Force, Army and Navy. The activities in each of these components can be further divided in four sub-components namely administration, intelligence, operations and training and supply. The applications of modern warfare techniques in each of the components of military organisations require expertise knowledge in respective fields.

ii. **Business Functions :**

   Each component in a business unit is an independent unit having their own goals. For example: production department minimises the cost of production but maximises output. Marketing department maximises the output, but minimises cost of unit sales. Finance department tries to optimise the capital investment and personnel department

iii. **Planning:**

In modern times, it has become necessary for every government to have careful planning, for economic development of the country. OR techniques can be fruitfully applied to maximise the per capita income, with minimum sacrifice and time. A government can thus use OR for framing future economic and social policies.

iv. **Agriculture:**

With increase in population, there is a need to increase agriculture output. But this cannot be done arbitrarily. There are several restrictions.

v. In Industry:

The system of modern industries is so complex that the optimum point of operation in its various components cannot be intuitively judged by an individual. The business environment is always changing and any decision useful at one time may not be so good some time later. There is always a need to check the validity of decisions continuously against the situations.

vi. **In Hospitals:**

OR methods can solve waiting problems in out-patient department of big hospitals and administrative problems of the hospital organisations. Techniques such as queuing theory helps in determining the number work stations required, for example, number of petrol pumps and attendants required in a petrol bunk. Numbers of out patient counters required in a hospital and so on based on estimated arrival of service seekers and time required to provide the service

vii. **In Transport:**

You can apply different OR methods to regulate the arrival of trains and processing times minimise the passengers waiting time and reduce congestion, formulate suitable transportation policy, thereby reducing the costs and time of trans-shipment.

viii. **Research and Development:**

You can apply OR methodologies in the field of R&D for several purposes, that as control and plan product introductions.

1.4 PHASES OF OR

Any business problem that needs the intervention of operations research can be solved using various steps. These steps rather phase help in identifying the nature of business problems and the appropriate tools or techniques required to solve the given problem. In many cases the data provided may not be sufficient as such more data can be sought on a given issue. The collected is used to formulate the objective function which shows clearly the objective of solving the problem Operation research also deals with dynamic programming where decisions have to be taken in a dynamic environment. In such cases data collection, analysis may become redundant. The various phases in solving the problems using operations research are

1. ***Judgment Phase:*** In this first phase the problem is identified that are encountered in the real life situations. The problem is structured in such way that the solution should support the organization objective by using proper judgment. The problem would be structured to facilitate the decision maker with all the possible information.
2. ***Research phases:*** On this second phase, further data is collected on the basis of the objective and given data. Hypothesis is framed and tested. The data collected is analyzed and verified. Suitable assumptions may be made wherever required.
3. ***Action Phase:*** In this last phase suitable recommendation are made for possible solution that has been arrived by solving the problem. Before implementing solution as suggested by the relevant OR model one may have to check the compatibility of the solution with the environment constraints and such other qualitative issues.

1.5 APPLICATIONS OF OR

Operations research has many applications on various fields in business management.

1. It is used while making decision about product to be produced and competitive strategies.
2. In scheduling salesman activities such time, territory, frequency of visit
3. In deciding type of promotion strategies to be employed
4. In forecasting the business direction
5. In deciding the price of the product with reference to competitor
6. In developing the framework for market research
7. In deciding the product mix and product proportioning
8. In production planning and sequencing and scheduling

9. In transportation ware housing and physical distribution
10. In material handling facility planning
11. In assemble line balancing
12. In maintaining of machines and replacing
13. In project planning and scheduling
14. In designing queuing system for serving customers
15. In maintain right size of inventory
16. In planning profit and selecting optimum dividend policy
17. In portfolio analysis i.e. product portfolio and investment portfolio
18. In determining optimum organizing structure

Apart from these OR also has applications in the field of defence, public system such as government and industrial applications.

1.6 LIMITATIONS OF OPERATION RESEARCH

Operations Research has number of applications; similarly it also has certain limitations. These limitations are mostly related to the model building and money and time factors problems involved in its application. Some of them are as given below:

i) ***Distance between O.R. specialist and Manager***

Operations Researchers job needs a mathematician or statistician, who might not be aware of the business problems. Similarly, a manager is unable to understand the complex nature of Operations Research. Thus there is a big gap between the two personnel.

ii) ***Magnitude of Calculations***

The aim of the O.R. is to find out optimal solution taking into consideration all the factors. In this modern world these factors are enormous and expressing them in quantitative model and establishing relationships among these require voluminous calculations, which can be handled only by machines.

iii) ***Money and Time Costs***

The basic data are subjected to frequent changes, incorporating these changes into the operations research models is very expensive. However, a fairly good solution at present may be more desirable than a perfect operations research solution available in future or after some time.

iv) Non-quantifiable Factors

When all the factors related to a problem can be quantifiable only then operations research provides solution otherwise not. The non-quantifiable factors are not incorporated in O.R. models. Importantly O.R. models do not take into account emotional factors or qualitative factors.

v) Implementation

Once the decision has been taken it should be implemented. The implementation of decisions is a delicate task. This task must take into account the complexities of human relations and behavior and in some times only the psychological factors.

1.7 SUMMARY

Operations Research is relatively a new discipline, which originated in World War II, and became very popular throughout the world. India is one of the few first countries in the world who started using operations research. Operations Research is used successfully not only in military/army operations but also in business, government and industry. Now a day's operations research is almost used in all the fields. Proposing a definition to the operations research is a difficult one, because its boundary and content are not fixed. The tools for operations search is provided from the subject's viz. economics, engineering, mathematics, statistics, psychology, etc., which helps to choose possible alternative courses of action. The operations research tool/techniques include linear programming, non-linear programming, dynamic programming, integer programming, Markov process, queuing theory, etc. Operations Research has a number of applications. Similarly it has a number of limitations, which is basically related to the time, money, and the problem involves in the model building. Day-by day operations research is gaining more and more acceptance because it improves decision making effectiveness of the managers. Almost all the areas of business use the operations research for decision making.

1.8 KEY WORDS

Operations Research

Dynamic programming

Optimization

Objective functions

Constraints

1.9 SELF ASSESSMENT QUESTIONS

1. Define operations research and explain the concept of operations research
2. Discuss the limitations of OR
3. Describe the scope and applications of OR in the present context
4. Outline the phases of OR
5. Give an account on various models of OR

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UNIT 2 : GAME THEORY

STRUCTURE

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2.3 The Maximin or Minimax Principle

2.4 Game without saddle Point

2.5 Minimax or Maximin Principle for mixed strategy Games

2.6 Principle of Dominance

2.7 2X2 mixed strategy using arithmetical method

2.8 Summary

2.9 Key Words

2.10 Self Assessment Questions

2.11 References

2.0 OBJECTIVES

After reading this unit you should be able to:

* Interpret the meaning of Game theory;
* Able to solve 2X2 matrix problem;
* Identify Dominance property and
* Able to solve 2 X m, n X 2 Problems graphically

2.1 INTRODUCTION

Game theory is a branch of applied mathematics and economics that studies situations where players choose different actions in an attempt to maximize their returns. First developed as a tool for understanding economic behavior and then by the RAND Corporation to define nuclear strategies, game theory is now used in many diverse academic fields, ranging from biology and psychology to sociology and philosophy. Beginning in the 1970s, game theory has been applied to animal behavior, including species' development by natural selection. Because of games like the prisoner's dilemma, in which rational self-interest hurts everyone, game theory has been used in political science, ethics and philosophy. Finally, game theory has recently drawn attention from computer scientists because of its use in artificial intelligence and cybernetics.

Although similar to decision theory, game theory studies decisions that are made in an environment where various players interact. In other words, game theory studies choice of optimal behaviour when costs and benefits of each option are not fixed, but depend upon the choices of other individuals.

2.2 GAME THEORY

Game theory is a branch of mathematical analysis developed to study decision making in conflict situations. Such a situation exists when two or more decision makers who have different objectives act on the same system or share the same resources. For example, companies selling same product. There are two person and multiperson games. Game theory provides a mathematical process for selecting an **optimum strategy** (that is. an optimum & decision or a sequence of decisions) to of an opponent who has a strategy of his own. The **games** studied by game theory are well-defined mathematical objects. A game consists of a set of players, a set of moves (or strategies) available to those players, and a specification of payoffs (result) for each combination of strategies.

A general theory of rational behaviour for situations in which (1) two (two- o9 person games) or more (multi-person games) decision makers (players) have-available to them (2) a finite number of courses of action (plays) each leading to (3) a well defined outcome or end with gains and losses expressed in terms of numerical payoffs associated with each combination of courses of action and for each decision maker. The decision makers have perfect knowledge of the rules of the game, i.e., (1), (2) and (3) but no knowledge about the opponents' moves and are rational in the sense of making decisions that optimize their individual gains. The matrix of payoffs can represent various conflicts. In a zero-sum game one person wins what the other loses. In other situations. gains and losses may be unequally distributed which allows the representation of numerous competitive and conflict situations. The theory proposes several solutions, e.g., in a minimax strategy each participants minimizes the maximum loss the other can impose on him, a mixed strategy involves probabilistic choices. Experiments with such games revealed conditions for cooperation, defection and the persistence of conflict. The theory and some of the results have found applications in economics, management science, bargaining and conflict resolution among many areas of interest.

2.2.1 History of game theory

The first known discussion of game theory occurred in a letter written by James Waldegrave in 1713. In this letter, Waldegrave provides a minimax mixed strategy solution to a two-person version of the card game. It was not until the publication of Antoine Augustia Cournot's *Researches into the Mathematical Principles of the Theory of Wealth* in 1838 that a general game theoretic analysis was pursued. In this work Cournot considers a duopoly and presents a solution that is a restricted version of the Nash equilibrium.

Although Cournot's analysis is more general than Waldegrave's, game theory did not really exist as a unique field until John Von Neumann published a series of papers in 1928. While the French mathematician Borel did some earlier work on games, Von Neumann can rightfully be credited as the inventor of game theory. Von Neumann was a brilliant mathematician whose work was far-reaching from set theory to his calculations that were key to development of both the Atom and Hydrogen bombs and finally to his work developing

Neumann and Oskar Morgenstern. This profound work contains the method for finding optimal solutions for two-person zero-sum games. During this time period, work on game theory was primarily focused on cooperative game theory, which analyzes optimal strategies for groups of individuals, presuming that they can enforce agreements between them about proper strategies.

In 2005, game theorists Thomas Schelling and Robert Aumann won the Bank of Sweden Prize in Economic Sciences. Schelling worked on dynamic models, early examples of evolutionary game theory. Aumann contributed more to the equilibrium school, developing an equilibrium coarsening correlated equilibrium and developing extensive analysis of the assumption of common knowledge.

2.2.2 Assumptions of Game theory

In game theory one usually makes the following assumptions:

(1) Each decision maker ["PLAYER"] has available to him two or more well-specified choices or sequences of choices (called "PLAYS").

(2) Every possible combination of plays available to the players leads to a well-defined end-state (win, loss, or draw) that terminates the game.

(3) A specified payoff for each player is associated with each end-state [ZERO-SUM game] means that the sum of payoffs to all players is zero in each end-statej.

(4) Each decision maker has perfect knowledge of the game and of his opposition; that is, he knows in full detail the rules of the game as well as the payoffs of all other players.

(5) All decision makers are rational; that is, each player, given two alternatives, will select the one that yields him the greater payoff.

(6) 'The last two assumptions, in particular, restrict the application of game theory in real-world conflict situations. Nonetheless, game theory has provided a means for analyzing many problems of interest in economics, management science, and other fields.

2.2.3 Uses of game theory

Games in one form or another are widely used in many different academic disciplines.

- Economics and business
- Applications in Biology

- Computer science and logic
- Political science
- Philosophy
- Sociology

2.2.4 Terminologies in Game theory

Player: The competitors in the game are known as players

Strategy: A strategy for a player is defined as a set of rules or alternative course of action available to him in advance by which player decides the course of action he should adopt. Strategy can be of two type:

a) Pure strategy: If the players select the same strategy each time then it is referred to as pure strategy. In this case each player knows exactly what other player is going to do. The objective of the players is to maximize gains or minimize losses.

b) Mixed strategy: When the players use a combination of strategies and each player always keep guessing as to which course of action is to be selected by thee other player at a particular occasion then this is known as mixed strategy.

c) Optimum strategy:A course of action or play which puts the player in the most preferred position, irrespective of the strategy of his competitors is called an optimum strategy.

d) Value of the game:It is the expected payoff of play when all the player s of the game follow their optimum strategies. The game is called fair if the value of the game is zero and unfair if it is non zero

e) Payoff Matrix:When the player select their particular strategies the payoff (Gains Or losses) can be represented in the form of a matrix called payoff matrix.

Let player A have m strategies A. , A_2, ... A_m & Player B have n strategies B_1 B_2.....B_n. The payoff matrix is written in terms of A i.e Positive values reflect gains to A & negative values reflect loss to A. Let *a;* be the payoff which player gains from B if player A chooses strategy Ai & Player B chooses strategy Bj. Then the pay off matrix is

Player B

		B1	B2	....Bj........	Bn
	AI	a11	a12	a1j	a1n
Player A	A2	a21	a22	a2j	a2n
	.				
	Ai	ai1	ai2	aij	ain
	An	an1	an2	anj	ann

2.3 THE MAXIMIN OR MINIMAX PRINCIPLE

For player A minimum Value in each row represents the gain (pay off) to him, if he chooses his particular strategy, They are written next to the matrix as row minima, He will then select the strategy that maximizes his minimum gains.

The choice of player A is called maximin value and the corresponding loss is the maximin value of the game. It is denoted by V

Player B minimizes his maximum losses. The maximum value in each column represents the maximum loss to him if he chooses his particular strategy. These are written in the matrix by column maxima. He will then select the strategy that minimizes his maximum loss. The choice of player B is called minimax value and the corresponding loss is the minimax value of the game.

It is denoted by V

By using this theorem, it is easy to determine a saddle point of a matrix.

Rules for determining saddle point

1. Select the minimum element of each row of the payoff matrix
2. Select the greatest element of each column of the payoff matrix
3. If the same element has been identified as row minima as well as column maxima then it is the saddle point.

2.0 OBJECTIVES

After reading this unit you should be able to:

* Interpret the meaning of Game theory;
* Able to solve 2X2 matrix problem;
* Identify Dominance property and
* Able to solve 2 X m, n X 2 Problems graphically

2.1 INTRODUCTION

Game theory is a branch of applied mathematics and economics that studies situations where players choose different actions in an attempt to maximize their returns. First developed as a tool for understanding economic behavior and then by the RAND Corporation to define nuclear strategies, game theory is now used in many diverse academic fields, ranging from biology and psychology to sociology and philosophy. Beginning in the 1970s, game theory has been applied to animal behavior, including species' development by natural selection. Because of games like the prisoner's dilemma, in which rational self-interest hurts everyone, game theory has been used in political science, ethics and philosophy. Finally, game theory has recently drawn attention from computer scientists because of its use in artificial intelligence and cybernetics.

Although similar to decision theory, game theory studies decisions that are made in an environment where various players interact. In other words, game theory studies choice of optimal behaviour when costs and benefits of each option are not fixed, but depend upon the choices of other individuals.

2.2 GAME THEORY

Game theory is a branch of mathematical analysis developed to study decision making in conflict situations. Such a situation exists when two or more decision makers who have different objectives act on the same system or share the same resources. For example, companies selling same product. There are two person and multiperson games. Game theory provides a mathematical process for selecting an **optimum strategy** (that is. an optimum & decision or a sequence of decisions) to of an opponent who has a strategy of his own. The **games** studied by game theory are well-defined mathematical objects. A game consists of a

2.4 GAMES WITHOUT SADDLE POINT

ALGEBRAIC METHOD

If the games do not have a saddle point, such problems can be solved in either in algebraic or arithmetical method. Consider the following example.

1. Determine the optimum strategy and the values of the game for the following payoff matrix O

		B: H	B: T
A	H	2	-1
	T	-1	0

The pay off matrix do not have any saddle point, Let player *A* plays H with probability x & T with probability with 1-x so that

$x + (1-x) = 1$

If Player B play H all the time Then A expected gain will be

E(A,H)- x.2 + (1-x) -1 = 3x-1

Similarly if Player B play T all the time then A 's expected gain will be

E(A,T) = x.(-l) + (1-x) 0 = -x

The best strategy for A is naturally the one which gives equal gain whether player B selects H or T. Then

3x-l = -x

Or 4x=i or x= ¼

The value of the game is 2X ¼ + ¾ X -1 = ¼

Similarly applying the same strategy for B, we get

Y= ¼

The solution is

1. The player A should play H or T with probability ¼ & ¾ respectively. The optimal strategy for A is { ¼, ¾ }
2. The player B should play H or T with probability ¼ & ¾ respectively. The optimal strategy for B is { ¼ , ¾ }
3. The expected value of the game for A is -1/4

2.5 MINIMAX OR MIXIMIN PRINCIPLE FOR MIXED STRATEGY GAMES

If a game does not have a saddle point, two players cannot use the maximin -minimax (pure) strategies as their optimal strategies. Hence the concept of mixed strategy i,e instead of selecting pure strategies only, each

The pay of matrix is as below

B

		Y,	Y_2	YJ	Y_n
		1	2	j	n
x,	1	V_{11}	V_{12}	$V._{1j}$	V_{1n} .
X2	2	V_{21}	V_{21}	$'v_{2j}$	V_{21}
X;	i	Vil	Vi2	Vij	Vjn
Xm	m	V_{m1}	V_{m2}	V_{nj}	v_{mn}

The lower value of game i.e., selected by players

$$V= \max_{xi} [\min[\sum_{i=1}^{m} Vi,Xi, \sum_{i=1}^{m} vi2Xi \ldots\ldots \sum_{i=1}^{m} Vi,Xi]]$$

Similarly the player B chooses

$$Yi(Yi>0, \sum_{j=1}^{n} Yi =1)$$

Which given the upper value

$$V= \min_{y_i} [man[\sum^{n} Vj,Xj, \sum^{n} vj2Xj \ldots\ldots \sum^{n} Vj,Xj]]$$

2.6 PRINCIPLE OF DOMINANCE

If the No. of strategic alternatives available for a player is more than 2 or 3 , then it is difficult to obtain value of game and also saddle point. In that case Dominance theory is used.

2.6.1 Inferior & Superior Strategies

Consider two strategies a & b for player A whose pay off are given by (a_b a_2, a_3....an) & (bj, b_2, b_3... .b_n), If $a_;$ > b; then a is said to be the superior strategy and b is said to be the inferior strategies.

Inferior strategies can be removed from the given pay off matrix so that a smaller pay off matrix is obtained.

2.6.2 Problem :

1. Solve this problem using dominance

$$
\begin{array}{cc}
 & B \\
A & \begin{pmatrix} -4 & 6 & 3 \\ -3 & -3 & 4 \\ 2 & -3 & 4 \end{pmatrix}
\end{array}
$$

Delete the columns which have higher or equal values when compared to the corresponding elements of another column.

Similarly delete the rows which have lower or equal values when compared to the corresponding elements of another row.

In this case columns 3 is dominated by column 1 . So neglect that column

$$
\begin{pmatrix} -4 & 6 \\ -3 & -3 \\ 2 & -3 \end{pmatrix}
$$

Now Row 2 is inferior compared to Row 3, So neglect row 2,

$$
\begin{pmatrix} -4 & 6 \\ 2 & -3 \end{pmatrix}
$$

Now Calculate row minima and column maxima

$$\begin{pmatrix} -4 & 6 \\ 2 & -3 \end{pmatrix} \begin{matrix} -4 \\ -3 \end{matrix}$$

$$\begin{matrix} 2 & 6 \end{matrix}$$

There is no saddle points so Let us use mixed strategy.

$V= -4_{x1}+2_{x2}$

$V = 6_{xl} + -3_{x2}$

$X_1 + x2 = 1$

i.e. $4x1 + 2_{x2} = v$ & $6x1 - 3x2= V$

$-4x\ 1 + 2_{x2} = 6_{xl} - 3_{x2}$

$10X1 = 5X2 \Rightarrow x1=0.5x2$

$x1 + x2 = 1$

$0.5\ x2 +x2 =1$

$1.5x2 = 1$

$= > X_2 = 2/3$

$= > Xi=l/3$

The player A choose mixed strategy

(1/3,0,2/3) •

Similarly

The player B choose mixed strategy

(3/5,2/5,0)

Value of the game is

$6x1-3x2$

$= 6 \times 1/3 - 3 \times 2/3$

$= 2- 2 =0$

2.7 2X2 MIXED STRATEGY USING ARITHMETICAL METHOD

A 2 X 2 matrix game with out saddle point can also be solved fusing arithmetical method ,.

Consider the following example

1. Solve the following problem for mixed strategy using mathematical method

	H	T
H	8	-3
T	-3	1

Solution:

Step - 1 : Try to obtain saddle point

B

H T

A

8	-3	4	4/(11+4) = 4/15
-3	1	11	11/(11+4) = 11/15
.3	1		11/15
4/15	11/15		

There is no saddle point

Take the difference between two numbers, in column I & put it under column II

i.e., 8-(-3) =11

Take the difference between two Numbers in column II & put it under column I

i.e., -3-1 = 4 (Neglect Sign)

Similarly

Take the difference between values of row 1 & put it next to row II

Take the difference between values of row 2 & put it next to row I

-3-1= 4 (neglect sign)

So,

The player A must use strategy H with probability 4/(4+l 1) = 4/15 and strategy T with 11/15 while player B use strategy H with probability 4/15 and strategy T with 11/15,

Let

B Play H

Then value of the game is

$$V = \frac{4x8 + 11x(-3)}{15} = \frac{-1}{15}$$

B Plays T

$$V = \frac{4x(-3) + 11x1}{15} = \frac{-1}{15}$$

A plays H

$$V = Rs. \frac{4x8 + 11x(-3)}{15} = \frac{-1}{15}$$

A plays T

$$V = Rs. \frac{4x(-3) + 11x1}{15} = \frac{-1}{15}$$

3. Use Dominance to solve the following problem

	A	B	C	D	E	F
A	0	0	0	0	0	0
B	4	2	0	2	1	1
C	4	3	1	3	2	2
D	4	3	7	-5	1	2
E	4	3	4	-1	2	2
F	4	3	3	-2	2	2

Column E is superior to column A, B, Row C is superior then A & B. Now the reduced matrix.

	C	D	E	F
C	1	2	2	2
D	7	-5	1	2
E	4	-1	2	2
F	3	-2	2	2

Column E dominates column F

	C	D	E
C	1	3	2
D	7	-5	1
E	4	-1	2
F	3	-2	2

Row E dominates row F

	C	D	E
C	1	3	2 •
D	7	-5 -	i
E	4	-1	2

Now take the avg. of player B , C & D strategy.

$$\frac{1+3}{2} \quad \frac{7-5}{2} \quad \frac{4-1}{2}$$

(2, 1, 3/2)

Superior to column E

Strategy E, So E can be eliminated

Similarly

The avg. C & D for player A is

$$\frac{1-7}{2} \quad \frac{3-5}{2}$$

(4, -1)

Which is equal to row E which can be eliminated A

Finally we land up with

	C	D
C	1	3
D	7	-5 -

For a $1X_1 + 7X_2 = V$ & $3X_1 - 5X_2 = V$

$= > X_1 + 7X_2 = 3X_1 - 5X_2.$

$= > 2X_1 = 12X_2.$

$X_1 = 6X_2.$

Substituting it is

$X_1 + X_2 = 1$

$= > 6X_2 + X_2 = 1 = > 7X_2 = 1$

$= > X_2 = 1/7$

$= > X_1 = 6/7$

For player A strategy is (0, 0, 6/7, 1/7, 0, 0)

Similarly for player B

$Y_1 + 3Y_2 = V$ & $7Y1 - 5Y_2 = V$

Solving which we get (0,0,4/7,3/7,0,0)

2.8 SUMMARY

Game theory is a type of decision theory in which one's choice of action is determined after taking into consideration ail possible alternatives available to an opponent playing the same game

The game theory is capable of analyzing single competitive situation. However there is great gap between what theory can handle and actual situation.

2.9 KEY WORDS

Game Theory

Strategy

Saddle point

Maximin or Minimax

Dominance

2.10 SELF ASSESSMENT QUESTIONS

1. Solve the following game

	1	2	3
1	-3	-2	6
2	2	0	2
3	5	-2	-4

2. Solve the following problem both arithmetically & algebraically.

		P2	
		I	II
P1	I	1	3
	II	4	2

3.

	A	B	C	D	E
A	4	4	2	-4	-6
B	8	6	8	-4	0
C	10	2	4	10	12

4. Solve this problem

I	2	4
II	2	3
III	3	2
IV	-2	6

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UNIT 3 : INVENTORY CONTROL

STRUCTURE

3.0 Objectives
3.1 Introduction
3.2 Terminologies used in Inventory Management
3.3 Categories of Inventory
3.4 Reasons for Carrying Inventory
3.5 Inventory Management
3.6 Cost Associated with Inventory
3.7 Inventory Models
3.8 Safety Stocks
3.9 Inventory Control Methods
3.10 Zero Inventory Systems
3.11 Problems on Inventory Management
3.12 Summary
3.13 Key Words
3.14 Self Assessment Questions
3.15 Reference

3.0 OBJECTIVES

After studying this unit you should be able to:

* Define inventory management;
* Asses the need to carry inventory;
* Analyze various models of inventory and
* Solve problems relating to inventory management.

3.1 INTRODUCTION

Inventory is a common phenomenon is business. The manufacturing industry has to have inventory to ensure smooth flow of production. An Inventory may be defined as a stock of idle tangible resources of any kind having an economic value. The inventories can be in the form of raw materials, semi-finished goods or finished product to be delivered to the customer. These could even be the human resources such as available unused labour or financial resources such as working capital and so on. For many organisations, inventories may be 30 to 70 per cent of the total assets. It varies from organisation to organisation. It is the level of inventory that matters for any organisation as capital investment is tied up in these resources.

Since it is blocking the working capital, which is so costly, it is not desirable to have - inventory. The inventory requires holding and maintenance or preservation cost. It carries the risk, spoilage, leakage or obsolescence. The cost of keeping inventory may be very high. Hence it is important to have a tight control over the level of inventory build up. It is a necessary evil, a must to keep uncertainty away in order to have business going, but should be kept only to the extent of minimum desired.

Controlling is a process by which the change in the system is modified to maintain the system an optimal performance level. The inventory control is highly desirable in any organization. Excess inventory on one hand results in blocking of working capital where the fewer inventories may affect the production on the other. Few of the examples of inventory in Product/Service industry are given be

Factory : Raw materials, parts and components, semi-finished inventory and finished goods.

Bank : Cash reserve tellers.

Hospitals : Number of beds, specialised personnel and stocks of drugs.

Airline : Aircraft seat miles per route, parts for repairs of aircrafts, stewards and other specialist persons for repair and maintenance.

3.2 TERMINOLOGIES USED IN INVENTORY MANAGEMENT

1) ***Demand:*** In order to decide on optimum level of inventory and its control policy, customers requirement in terms of its size (number of items required), the rate (how many items are required and the pattern (whether continuously increasing or decreasing and at what rate, or whether a seasonal demand) need to be collated. Hence demands can be deterministic or probabilistic. Deterministic demands are those types of demands, which can be predicted or known with certainty with a definite time frame whereas probabilistic demands are those, which cannot be known in either form i.e., neither its quantity, nor time schedule nor the pattern be predicted.

2) ***Order Cycle:*** It is the time period between two successive orders placed to meet the demand; it needs to be established, if there is a set pattern in placing the orders. This is possible for such situations demands are known with a definite pattern and are constantly reviewed. But for situations like scale projects, environmental factor take toll of the pattern and order cycle may be difficult to adhere to. Then items are ordered as and w hen required.

3) ***Time Horizon:*** It is the period over which the inventory level will be controlled. It can be finite or infinite depending on the nature of demand.

4) ***Lead Time:*** It is the time elapsed between the time of ordering the item and its actual receipt /place of requirement. Lead time plays a very important role in the inventory control policy. The cost associated with inventory are largely dependent on the lead time, which can be constant, variable, deterministic or probabilistic. The best situation can be JIT (Just-in-time) i.e., the situation of zero lead If the material can be received when item is required, the order need not be placed in advance and may be no requirement for carrying cost. Another good situation exists when lead time is deterministic. All functions of inventory are then under control with definite system. But when lead time ain, then ordering and carrying costs can disturb the balance of inventory control.

5) ***safety Stock or Buffer Stock:*** It is the level of inventory kept procured when either the lead time is uncertain or the demand is critical and shortage cost may be high. This inventory is planned to meet the demand during uncertain supply period or else to cater for sudden spurt in demand for a short duration.

6) ***Re-order Quantity:*** It is the quantity of items ordered to replenish the exhausted or utilised inventory with a comprehensive inventory policy. It should be the Economic

7) ***Re-order Level:*** It is the level of stock inventory at which it is decided to replenish the stock. It is connected with the lead time, such that the item should be received just at a time when the stock level is at the minimum desired level. In quite a few cases, when safety of buffer stock is planned, the ~e-order level) should cater for the level of consumption of inventory just sufficient to reach Safety stock level during the lead time.

3.3 CATEGORIES OF INVENTORY

There are inventories maintained for various purposes. Since inventory normally consumes and blocks a substantial amount of working capital, it is imperative for all managers to understand the types and purpose of inventory. Also there has to be a system of knowing the existence of inventory, otherwise may not come to the notice at the right time of taking crucial decisions. In any organisation, inventories can be classified under various categories, all of these may or may not exist at any one time

Movement or Process Inventory: These inventories exist because transportation takes time in reaching the inventory to the point of consumption. These could be in the form of finished goods inventory in transit from manufacturing base to the market or the raw materials in transit from source to the place of utilization.

1. ***Buffer Inventory:*** It is kept in the form of safety stock to cater for the uncertainty of demand or supply. Future demands are normally forecasted, but forecasts may not depict changed situation or uncertain environment, thereby demands changing without notice. Shortage cost may be very large, when customer demands are not met for want of lag-behind production due to non-availability of raw material, or finished good inventory at sufficient level in case of spurt in seasonal demand. Buffer stocks also act as safety against uncertain lead time or delays due to natural calamities.
2. ***Seasonal Inventories:*** Seasonal inventories are kept to cater for higher demand during a particular period of the year. Instead of increasing the production level, which may be quite capital-intensive to provision new facilities, it may be economical to manufacture product during the lean or off-season, when demands are low, and production facility can be gainfully utilised to build up stock for a higher-demand period. Here a balance of capital investment and cost of its procurement with that of cost of carrying seasonal inventories has to be judiciously worked out. It can also be called 'Anticipation' inventory as items like air conditioners, air coolers, crackers, umbrellas and raincoats are required against very specific sudden demand and for a very short duration, but in large quantity.

3. ***Decoupling Inventory:*** These inventories are necessary to reduce dependence of various stages of production. These may be raw materials, work in process inventories or finished goods, under the conditions when certain production facility may not match with the system due to poor condition of machine, machine breakdown or working efficiency differential of workers. This inventory is also required for after-sales service, where demands do not follow a pattern.

3.4 REASONS FOR CARRYING INVENTORY

Some of the important reasons for carrying inventory emerge out of the environmental conditions and can be summarised as follows:

1. Variation in production
2. Variable customer lead time
3. Uncertain vendor reliability in quality
4. Financial gains when prices are uncertain or fluctuate or else when quantity discounts are attractive.

3.5 INVENTORY MANAGEMENT

Inventory management is necessary in the follows areas:

1. Accounting for inventories to workout lead time, source order restrictions, receipt quality and time, audit and control of cash flow.
2. Operating constraints—these are considered for working out optimal inventory policy, limited merchandising and limited budgeting etc.
3. Planning and control
    a) What to buy and where from?
    b) When to buy and how much?
4. Measure of performance — for the purpose of satisfaction of forecasted demand.

3.6 COST ASSOCIATED WITH INVENTORY

There are following types of costs involved, while discussing and making decision on inventor

1. ***Ordering Costs:*** Inventory, when necessary, need be created and in order to achieve this have to go through certain administrative functions while ordering the items. The documentation cost of ordering organisations, the communication cost like telephones,

fax, postage etc. need be cons In addition, we have to cater for the transportation of items, inspection at vendors end or at the when received, the cost of receiving and then processing the payments etc.

2. ***Purchase Costs:*** The cost of purchasing, after due negotiations, per unit item is called the purchase cost. Discounts and price breaks can be secured during negotiations for reducing the cost, when large quantity is ordered.
3. ***Carrying Costs:*** The costs associates with holding inventories are called carrying The quantity of items actually held in stock only need be considered for such costs. Larger quantity for longer period would mean larger carrying costs. It includes the cost of storage, maintenance, deprecition cost of security and accounting and taxes paid for the inventories etc.
4. ***Shortage Costs:*** These are the costs incurred for not holding the inventory, when needed, the penalty for running out of stock and is due to the fact that the product is not available to the c when in demand. This cost would include the loss of opportunity and credibility of the organisation not meeting the demand of the customer. The loss of goodwill and reduction in further procurement business are important effects and hence shortage costs can be very large, still may not be quantifiable and very apparent. Certain effects are felt over a long period.

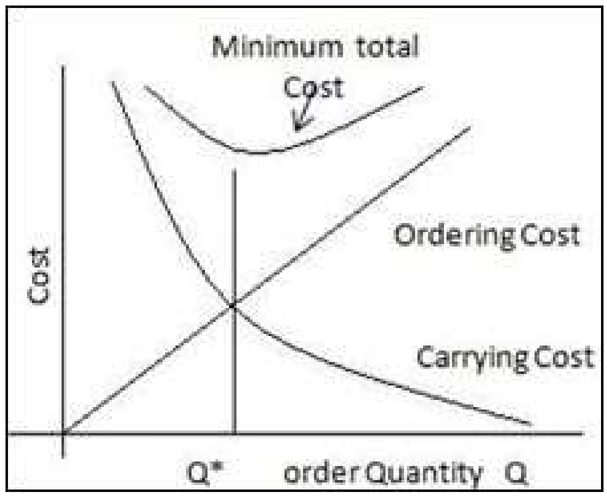
   There may be situations when customer can wait for the demand to materialise, but proc of such stocks would involve additional efforts of special order and emergency price paid for eventuality. This cost is worth the efforts, if good-will is to be saved. To minimise shortage, additional stock of appropriate level can be kept in the form of safety or Buffer Stocks, essential when items are of critical nature from the business point of view.
5. ***Salvage Costs:*** When the demand for an item ceases to exist, the decision to dispose off or sell the item may be taken or, if the item is in inventory and is deteriorating, it needs be either used sold at a discount. The cost normally gets associated with storage or holding cost and is not considered separately.

Therefore, in order to decide on the optimal inventory policy, as the costs described be considered dispassionately and level of inventories worked out based on a balanced view business situation.

Thus, Total Inventory Cost - Ordering Cost + Purchase Cost + Holding Cost + Shortage cost.

Hence when to order and how much to order becomes important inventory optimisation.

The relationship of various costs on the inventory level is shown in the Fig. 1: Point Q* indicates optimum ordering quantity for minimum total inventory costs.



3.7 INVENTORY MODELS

To meet various life situations, we need the consideration of relevant factors to determine invent models which help in an effective inventory control policy.

Model I — Economic Order Quantity Model with Uniform Demand

In this case, we keep a watch on the inventory, which is deterministic due to uniform demand le and we need developing a model, which caters for replenishment at a time, when lead time provides sufficient opportunity of replenishment. EOQ is the quantity required to be ordered to keep the u inventory cost, at the minimal level.

Let D = Demand (uniform and deterministic during lead time)

R = Rate of replenishment

L = Lead time

Cp = Unit purchase cost

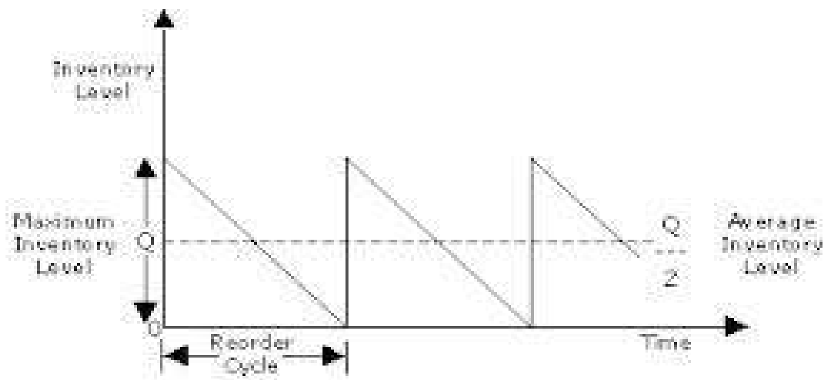
Ch = Unit holding cost

Co = Unit ordering cost

Cs = Shortage cost

Q = Decision variable ie., Quantity to be procured.

Let us draw a graphical representation of the situation of EOQ Model with uniform den



Referring to Fig. 2 above, maximum inventory level is taken as Q, with uniform consumption of inventory i.e., uniform demand. It is shown as a sloping straight time to reach z; at a uniform rate. Since the demand is uniform, the re-order cycle also become definite, inventory can be worked out as follows :

$$\frac{Maxlevel + Minlevel}{2} = \frac{Q+0}{2} = \frac{Q}{2} = \text{Average Inventory}$$

We have to find optimum level Q such that the total inventory costs are minimised.

(a) Ordering Cost = $\frac{D}{Q}Co$

(b) Carrying Cost = $\frac{Q}{2}Ch$

Hence Total Inventory Cost = $\frac{D}{Q}Co + \frac{Q}{2}Ch$

In this case, lead time being constant, ordering cost = carrying cost

$$\therefore \quad \frac{D}{Q}Co + \frac{Q}{2}Ch$$

$$\therefore \quad Q = \sqrt{\frac{2.D.Co}{Ch.}}$$

This model is known as WILSON LOT SIZE FORMULA.

Model II - Economic Order Quantity with Variable Demand

In this model formulation, all other assumptions remain the same as in the case of model I, except condition of variable demand. In that case, we have to specify the time period for the demand. By this assumption

Ordering cost = $\frac{D}{Q}Co$

Carrying cost = $\frac{Q}{2}Ch.T.$ (where T = Time period for the demand)

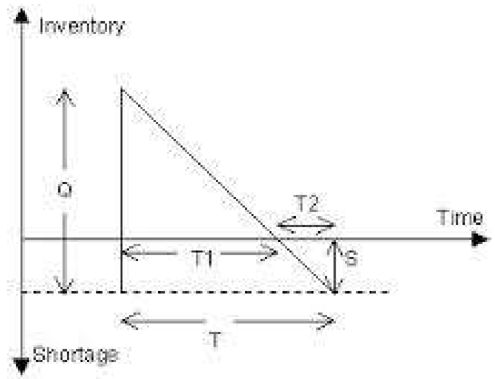
Total cost = $\frac{D}{Q}Co + \frac{Q}{2}Ch.T.$

and Q (EOQ) = $\sqrt{\frac{2DCo}{TCh.}}$

Total Optimal Inventory = $D \; Co \Big/ \sqrt{\frac{2DCo}{Ch.}} + \frac{1}{2}Ch.T.\sqrt{\frac{2DCo}{TCh.}} = \sqrt{\frac{2DCoCh}{T}}$

Model Ill - Economic Order Quantity when Shortages are Permitted

In this case, the change from Model I is that inventory levels are allowed to go below zero level and shortage is directly proportionate to the number of units short. The figure 3 indicates the situation



Shortages Permitted Model

Total inventory is Q, but it consists of Q_0 i.e., inventory and *s,* the demand during shortage period. The cycle time *T* has been accordingly marked

In this case, Total Cost = Ordering Cost + Carrying Cost + Shortage Cost

= Co + Ch + Cs. (where Cs - Shortage Cost)

Here Average Inventory = $\frac{Q_o}{Q}$

and Ordering Cost = $\frac{D}{Q}Co$

Carrying Cost = $\frac{Qo}{Q}t_2Ch$

$= \frac{Qo}{Q} . \frac{Qo}{Q}tCh$

$= \frac{Qo2}{2Q}.t.Ch$

Shortage Cost $= \frac{s}{2}t_1C_s$

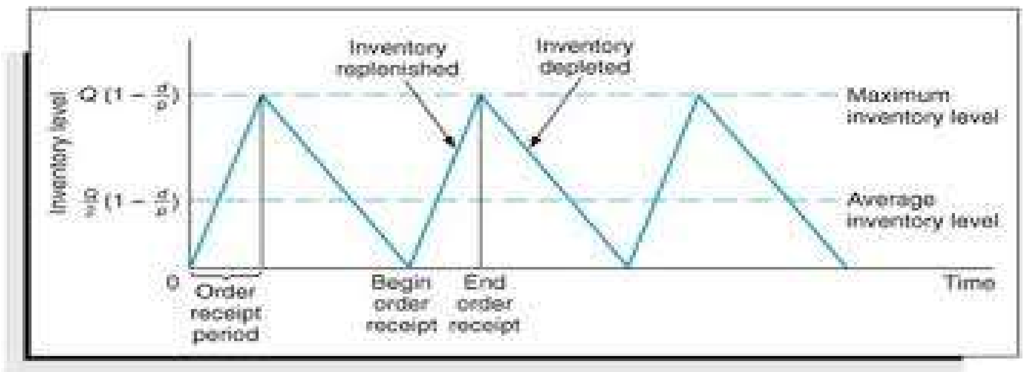
$= \frac{s}{2}\frac{s}{Q}.t.Cs$

$= \frac{s^2}{2Q}.t.Cs$

Total Inventory Cost $= \frac{D}{Q}Co + \frac{Q_0^{\ 2}}{2Q}tCh + \frac{s^2}{2Q}t.Cs$

Model IV - Economic Order Quantity with Gradual Replenishment

In this case, the replenishment of the inventory is not in one go, but gradual like the **or** the rate of replenishment and consumption, though, may not be the same. This is a practical situation because production consumes the inventory over a period of time and gradually. Hence inventory all at one time may not be a good policy. Gradual replenishment is much nearer requirement, though may not be good from vendor point of view. Such a situation of rep consumption is given in the figure below.



If Demand is D, which remains constant and uniform over a period of time, and replenishment is gradual as indicated above.

Let tp = production time and

rp = rate of production

Q = size of each production run pef cycle.

rd = demand rate

Then by normal convention of Co, Ch and D

$$O(EOQ) = \sqrt{\frac{2DCo}{Ch}\left(\frac{rp}{rp-rd}\right)}$$

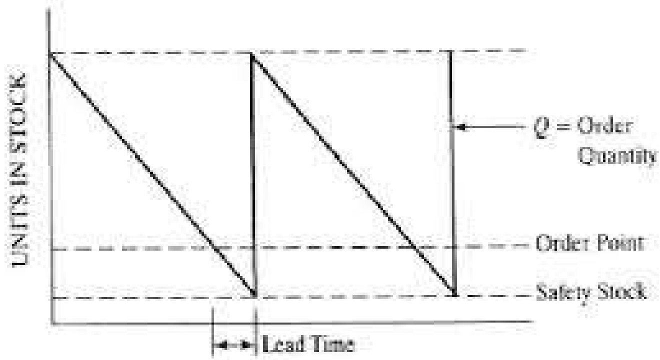
$$tp = \sqrt{\frac{2DCo}{Ch.rp(rp-rd)}}$$

Total Cost at Minimum inventory

$$= \sqrt{2DCoCh}\left(\frac{1-rd}{rp}\right)$$

3.8 SAFETY STOCK

In addition of the inventories required for normal functions of business, some inventories are needed to cater either sudden spurt of demand or uncertainty of usage or replenishment rate. When lead times are uncertain, we smooth out the production or meeting the demand from the safety or Buffer stock. It can understood from the figure 5 given below :



In order to work out the safety stock, we have to consider the effect of shortage, if the safety stock is not kept. The alternatives and the effect of shortage vary depending upon the situation. If items in the shortage are very critical for the operation, shortage cost can be very high. Loss of or loss of credibility is high cost to pay for the shortages. Loosing the market share even temporarily can be very devastating at times, when it is difficult to regain it. Even to cater for loss of production due to interruption in supplies either due to natural calamities or human created obstructions such as strikes etc. the buffer stock or safety stock becomes essential. Hence variability of either demand or time necessitates the stocks as safety stocks and can be worked out as follows :

Safety Stock = (Max. demand - Average demand during lead time x Lead time)

3.9 INVENTORY CONTROL METHODS

In any organisation, depending on the type of business, some inventory is maintained. When the number of items in inventory are large and large amount of money is needed to create such an inventory, it become the concern of management to have a proper control over its ordering, procurement, maintenance and consumption. Unless some effective method is evolved, the control process can be cumersome. The control is different for different type of inventory usage and can be classified as under.

1) The order quantity
2) The order frequency

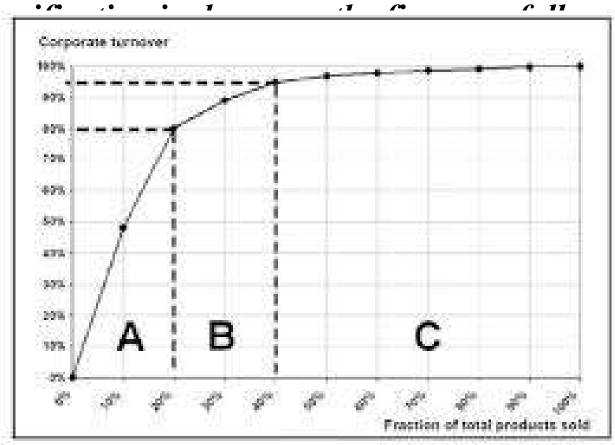
3) The time interval between successive reviews.

The most widely used method of inventory control is known as ABC analysis. It is often called as Always Better Control Analysis or Pareto Analysis. In the technique, the total inventory is categorized into three sub-heads and then proper attention is exercised for each sub-head.

A-B-C Analysis

In this analysis, the classification of existing inventory is based on annual consumption and the annual value of the items. Hence we obtain the quantity of inventory item consumed during the year and multiply it by the unit cost to obtain it annual usage cost. The items are then arranged in the descending order of such annual usage cost. The analysis is carried out by drawing a graph based on the c number of items and cumulative usage of consumption cost. Classification is done as follows.

The cl



Once ABC classification has been achieved, the policy for control can be formulated as follows

A-items : Very tight control, the items being of high value. This control need be a exercised at higher level of authority.

B-items : It requires moderate control by a person at middle level authority.

C-items : These being low value, but large number items, control can be exercised by grass-root level i.e., this control can be done by respective user departmental managers.

Other Inventory Control Methods

1) HML - The items are classified as per the unit price of items such as:
   H = High price
   M = Medium price
   L = Low price

2) VED - The classification is done based on the critically of items such as:
   V = Vital few
   E = Essential
   D = Desirable

3) FSN - Yet another control is exercised on items classifying them on the basis of consumption:
   F = Fast Moving items
   S = Slow moving items
   N = Non-moving items

4) SDE - This classification and control is resorted to on the basis of problem faced on their procurement such as:
   S = Scarce
   D = Difficult
   E = Easy to obtain

5) SOS - Here the control requirement is based on the periodicity of demand or nature of supplies:
   S = Seasonal
   OS = Off-seasonal

6) GOLF - The classification is based on the source of the inventory:
   G = Government supply
   O = Ordinarily available
   L = Local availability
   F = Foreign source of supply

3.10 ZERO-INVENTORY SYSTEMS

At times inventory holding or carrying cost are very excessive or there may be space restrictions on holding inventory. In such case, a concept of zero-inventory as adopted by Japanese may be worth following. In this ease, we attempt to bring down the inventory to almost zero or totally eliminate the inventory. The method can be practiced by asking for the materials or the products from the vendors locally available, keeping quality requirement in mind.

Another related or associated concept practised by Japanese is JIT (Just-in-Time). The Kanban system developed and used by them is a step in this direction. In this case, inventory in raw materials are obtained only when required and that too just to cater for the days requirement. The material is delivered to the work centre, when it is required. Hence raw materials, work in process as well as Finished goods inventories are minimal. It is good concept to reduce waste and also to save on storage space and cost of accounting and maintaining the inventory. The system is becoming popular in all the including India.

3.11 PROBLEMS ON INVENTORY MANAGEMENT

Problem 3.1 :

X Y Z company buys in lots of 500 boxes which is a 3 month supply. The cost per box is Rs.125 and the ordering cost is Rs. 150. The inventory cost is estimated at 20% of unit value.

i. What is the total annual cost of the existing inventory policy?

ii. How much money could be saved by employing the economic order quantity?

[C.A. *(Final), No.* 1983],

Solution :

Ordering Cost (Co) = Rs. 150

Quantity of one order = 500

Annual demand = 500 x $\frac{12}{3}$ = 2,000

Carrying Cost = 125 x 20% = Rs. 25

(i) Total annual Cost of existing inventory

$$= \frac{D}{Q}.Co + \frac{Q}{2}.Ch$$

$$= \frac{2000}{500} \times 150 + \frac{500}{2} \times 25 = Rs.6850$$

(ii) Economic order quantity

$$= \sqrt{\frac{2DCo}{Ch}}$$

$$Q(EOQ) = \sqrt{\frac{2 \times 2000\ X150}{25}}$$

= 155

Minimal Annual Cost $= \sqrt{2DCoCh.} \quad = \sqrt{2 \times 2000 \times 150 \times 25} = 3873$

Hence saving $= Rs.6850 - 3873 = Rs.2977$

Problem 3.2:

The demand of an item is uniform at a rate of 25 units per month. The fixed cost is Rs 30 each time a production is made. The production cost is Rs. 2 per item and the inventory carrying cost is 50 paisa per unit per month. If the shortage cost is Rs. 3 per item per month, determine how often to make a production run and of whit size?

Solution :

Given data indicates following values

D = 25 units

Ch = Re. 0.50 per item per month

Cs = Rs. 3 per item per month

Production Cost = Rs. 2 per item

Production Set up Cost = Rs. 30 per set up.

$$Optimal(EOQ) = \sqrt{\left(\frac{2DCo}{Ch}\frac{Ch + Cs}{Cs}\right)}$$

$$= \sqrt{\left(\frac{2 \times 25 \times 30}{0.5}\frac{0.5 + 3}{3}\right)}$$

$$= 60 units$$

$$Optimal\,time = \sqrt{\frac{EOQ}{D}}$$

$$= \sqrt{\frac{60}{25}}$$

2.4 Months

The annual demand of a product is 1,00,000 units. The rate of production is 2,00,000 units per.The set up cost per production run is Rs. 500 and the variable production cost of each item is Rs.10 The annual holding cost per unit is 20% of its value.

Find the optimum production lot size and the length of the production run.

Given data, D = 1,00,000; rp = 2,00,000, Ch = 0.2, rd = 1,00,000 and Co = 500.

Optimum production lot size is given by

$$= Q(EOQ)\sqrt{\frac{2DCo}{Ch}\left(\frac{rp}{rp-rd}\right)}$$

$$= \sqrt{\frac{2\times 1,00,000\times 500}{0.2}\left(\frac{2,00,000}{2,00,000-1,00,000}\right)}$$

=10,000 Units

$$\text{Length of the production run} = \frac{\text{Q}}{\text{rp}} = \frac{10,000}{2,00,000}$$

=0.05 year

<u>Problem 3:</u>

A contractor has to supply 10,000 paper cones per day to a textile unit. He finds that when he starts a production run, he can produce 25,000 paper cones per day. The cost of holding a paper cone in stock for one year is 2 paisa and set up cost of production run is Rs. 18. How frequently should the production run be made?

Solution:

Let us assume that the production runs on 300 day in a year

Given D = 10,000 x 300 = 30 Lakhs

Co = Rs. 18

Ch = 0.02 per unit per year

rp = 25,000

rd = 10,000

$$= Q(EOQ)\sqrt{\frac{2DCo}{Ch}\left(\frac{rp}{rp-rd}\right)}$$

$$= \sqrt{\frac{2\times 30,00,000\times 18}{0.02}\left(\frac{25,000}{25,000-10,000}\right)}$$

=95,000 Units

$$\therefore Frequency\ of\ production = \frac{95,000}{25,000} = 4\ Days$$

Problem 3.5 :

A company uses annually 50,000 units of an item costing Rs. 1.20. It operates 250 days in a year and procurement time is 10 days. Each order costs Rs. 45 and inventory carrying cost is 15% of the annual average inventory value. If the safety stock is 500 units, find EOQ, Re-order level. Minimum and maximum and average inventory levels.

Solution :

Given Co = Rs. 45

Ch = 0.15 of 1.10 = Rs. 0.18

D = 50,000

$$economic\,order\,quantity = \sqrt{\frac{2 \times 45 \times 50{,}000}{0.18.}} = 5{,}000\,units$$

$$Average\,daily\,demand = \frac{50000}{250} = 200\ units\,per\,day\,(for\,250\,days\,annual\,working)$$

Lead time=10 days

Average DDLF=20×10=2,000 units

Safety stock=500 units

ROL =Safety stock + average DDLF

=500+2000=2500 Units

Maximum Inventory = 500+ 5000 = 5500 units

Minimum inventory = Safety stock = 500 units

Average inventory = 500+1/2 EOQ = 3000 units.

3.12 SUMMARY

Inventory is essential to provide flexibility in operating system or organization. Inventory can be raw material, work in progress and finished product. In business maximum inventory has to be maintained to ensure proper flow of production. A huge amount of working capital is blocked in inventory. Hence it is highly desirable to maintain minimum inventory. This optimal amount of inventory can be determined using mathematical formulas. The data such as inventory carrying cost, ordering cost and so on should be collected to determine the economic order quantity.

3.13 KEY WORDS

Inventory

Safety Stock

ABC analysis

Economic Order Quantity

EOQ model with shortage

EOQ model with gradual replenishment

3.14 SELF ASSESSMENT QUESTIONS

1. A factory requires 1500 unit of an item per month, each costing Rs 27. The cost order is Rs 150 and inventory carrying charges work out to 20% of the average inventory. Find out the economic order quantity and number of orders per year. Would you accept a 2% price discount on a minimum supply quantity of Rs 1200 units. Compare the total costs in both the cases.
2. The annual requirements for a particular raw material are 2000 units costing Re 1 each to the manufacturer. The ordering cost Rs 10 per order and the carrying cost 16 % per annum of the average inventory value. Find the economic order quantity and the total inventory cost per annum.

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UNIT 4 : QUEUING THEORY

STRUCTURE

4.0 Objectives

4.1 Introduction

4.2 Queue

4.3 Benefits and Limitations of Waiting Line Theory

4.4 Types of Queuing models

4.5 Single – Channel Queuing Model

4.6 Multiple channel queuing Model

4.7 Summary

4.8 Key Words

4.9 Self Assessment Questions

4.10 References

4.0 OBJECTIVES

After studying this unit, you should be able to :

* Define queue;
* Analyze the benefits and limitation of waiting line theory;
* Compare varies queuing models and
* Solve problem of queuing models

4.1 INTRODUCTION

The study of waiting lines, called 'queuing theory', is one of the oldest and most widely used Operations research techniques. The first recognized effort to analyze queues was made by Danish Engineer, A.K. Erlang, in his attempts to eliminate bottlenecks created by telephone calls on switching circuits. Number of waiting time situation exists in industry. Here are just a few instances:

1. Computer programs are waiting to be processed in a computer.
2. Customers are waiting to be served at banks, hotels, Petrol bunks.
3. Parts are waiting to be processed at a manufacturing operation.
4. Machines are waiting to be repaired at a maintenance shop
5. Trucks are waiting to unload their cargo at a unloading dock.

In these and other waiting line situations, managers don't necessarily plan for waiting lines to form; rather, waiting lines are inevitable characteristic of these operations. In general queue is formed when either units requiring services commonly referred to as customers, wait for service or the service facilities, stand idle and wait for customers.

4.2 QUEUE

A queue refers to the customers waiting for service. This does not include the customers being served. Some operational situation allows a queue of any size to form: In others, the queue may not exceed a certain length. A queue is characterized by its maximum permissible size, which may be infinite or finite. In a sales department where the customer orders are received, there is no restriction on the number of orders that can come in so that queue of any size can form when there is no limit on its size, the permissible queue is said to be infinite.

In a petrol pump, the space for waiting of vehicles is usually limited. If a driver arrives queue is limited by the space for waiting. In many other situations an incoming customer may not enter the service system if a certain number of customers are already waiting even though additional waiting space is available. Queue size in this case is controlled by the attitude of the customers. When there is limit on the size, the permissible queue is said to be finite.

4.4 BENEFITS AND LIMITATIONS OF WAITING LINE THEORY

The effort of A.K.Erlang in 1909 to analyze telephone traffic congestion with the objective of meeting uncertain demand for services at Copenhagen telephone system which resulted in new theory (Queuing Waiting line theory) is valuable tool in business because many business problems can be characterized as arrival -departure congestion problems. Techniques of queuing, theory have been applied for a solution of a large number of problems such as ;

1. Scheduling of aircraft at landing and takeoff from a busy airports.
2. Scheduling of issue and return of tools by workmen from tool cribs in factories.
3. Scheduling of mechanical transport fleets
4. Schedule a distribution of scarce war material.
5. Scheduling of work and jobs in production control.
6. Minimization of congestion due to traffic delay at tool booths.
7. Scheduling of parts and components to assembly lines.
8. Decision regarding replacement of capital assets taking into consideration mortality curves, technological improvement and cost equations.
9. Routing and scheduling of salesmen and sales efforts.
10. Queuing theory attempts to formulate interpret and predict for purpose of better understanding the Queues and for the scope to introduce remedies such as adequate service with tolerate waiting.
11. Queuing theory provides models that are capable of influencing arrival pattern of customers or determine the most appropriate amount of service or number of service stations.
12. Queuing models essentially relates to study of behaviors of waiting lines via mathematical techniques utilizing concept of stochastic Process.

Limitations of queuing theory are as follows:

1. Queuing problems are generally complex in nature and cannot be easily solved. The element of uncertainty is there in almost all queuing situations.
2. In most of the cases, the observed distributions of service times and time between. arrival cannot be fitted by mathematical distribution as we generally assume while solving queuing problems.
3. In multi -channel queuing situations, the analysis becomes more difficult when the departure from one queue becomes the arrival of another queue.
4. Queuing models with variable arrival and / or service times are so complicated in nature that they cannot be handled by formulae and mathematical methods. In such situations, Monte Carlo method of simulation is better answer.

4.4 TYPES OF QUEUING MODELS

The basic queuing models can be classified into six categories using Kendall notation which in turn uses six parameters to define a model such (A/B/C):(a/b/c). The parameters of this notation are

A: Arrival rate distribution, i.e. number customers arriving at a service station per hour. Usually the arrival rates is assumed as poison rate of distribution.

B: Service rate distribution i.e number customers being served per hour

C: Number of service centers or servers available in a service station.

a: Service Discipline

b: Maximum number of customers permitted in the system

c: Sizes of calling source of the customers i.e number of possible customers who may enter system

Kendall Notation:

Model No 1: (M/M/1) : (GD/∞/∞) Poisson arrival rate, poison service rate, single server, General discipline, Infinite number of customer is permitted in the system, size of calling source is infinite

Model No 2: (M/M/C) : (GD/∞/∞) Poisson arrival rate, poison service rate, Multi server, General discipline, Infinite number of customer is permitted in the system, size of calling source is infinite

Model No 3: (M/M/1) : (GD/N/∞) Poisson arrival rate, poison service rate, single server, General discipline, Finite number of customer is permitted in the system, size of calling source is infinite

Model No 4: (M/M/C) : (GD/N/∞) Poisson arrival rate, poison service rate, multi server, General discipline, Finite number of customer is permitted in the system, size of calling source is infinite

Model No 5: (M/M/1) : (GD/N/N) Poisson arrival rate, poison service rate, single server, General discipline, Finite number of customer is permitted in the system, size of calling source is finite

Model No 6: (M/M/C) : (GD/N/N) Poisson arrival rate, poison service rate, multi server, General discipline, Finite number of customer is permitted in the system, size of calling source is finite

4.5 SINGLE – CHANNEL QUEUING MODEL (M/M/1) : (GD/∞/∞)

The single-channel, single phase model is one of the most widely used and simplest queuing model. It assumes that the followings seven conditions exist:

1. Arrivals are served on a 'First come first served' basis.
2. Every arrival waits to be served regardless of the length of the line.
3. Arrivals are independent of preceding arrivals, but the average umber of arrivals (the arrival rate) does not change over time.
4. Arrivals are described by a Poisson distribution and come from an infinite (or a very large) population.
5. Service times also vary from one customer to the next and are independent of one another, but their average rate is known,
6. Services times occur according to the negative exponential probability distribution.
7. The average service rate is greater than the average arrival rate.

4.5.1 Queuing Equations

The quantitative expressions developed for analyzing the single-channel waiting line are :

e = Mean or expected number of arrivals per time period (mean arrival rate)

i = Mean or expected number of items served per time period (mean service rate)

Probability of service being busy

$$n = \frac{e}{i}$$

$= \frac{\text{Average rate of arrival}}{\text{Average rate of service}}$

Using the assumptions of Poisson arrival and exponential service times, the following expressions define the characteristics of the single-channel waiting line, 1.

1. The probability that the service facility is idle.

   $$P_o = 1 - \frac{e}{i}$$

   Pn=probability that there are 'n' units in the system $\quad Pn = \frac{e}{i}$

2. Expected or average number of units in the system. (i.e queue + being serviced)

   En or Ls = i / (i - e)

3. Mean (expected or average) number of units in the queue waiting for service

   $$\text{Em or Lq} = \frac{i^2}{e(e-i)}$$

4. Mean (expected) waiting time in the system $\frac{1}{e-i}$

5. Mean (expected or average) time

   $$\frac{e}{i(i-e)}$$

6. Probability that the queue size is equal to k

7. Probability that the queue size is greater than K

8. Probability of exactly n number of customers in the system

Problem-1: Customer arrive at the ticket counter of a theater at the rate of 12 per hour .There is one clerk serving the customer at the rate of 30 per hour. Assuming the condition for use of single-channel queuing model, calculate:

1. The probability that there is no customer at the counter
2. The probability that there are more than 2 customer at the counter
3. The probability that there is no customer waiting to be served.

Solution:

Arrival Rate (e) = 12 per hour and service rate (i) = 30 per hour

1. Probability that there are no customers Po = 1 - e/i = 1- (12/30) = 0.6
2. Probability that there are more than 2 customer at the counter $(e / i)^k = (12/30)^3 =$ 0.064.
3. Probability no customer is waiting (at most one customer at the counter) = PO +P1= 0.6+0.6x0.4=0.84

Problem-2: A TV repairman finds that the time spent on his job has an exponential distribution wit mean 30 minutes. If he repairs sets in the order in which they come in and if the arrival of sets is approximately Poisson with a average rate of 10 per 8 hour day.

1. How many jobs are head of the set just brought in?
2. What is the repairman's expected idle time each day?

Solution :

Mean arrival Rate (e) = 10 / 8 per hour = 1.25 per hour

Mean service rate (i) =60 / 30 =2 per hour

1. Average number sets ahead of the one just brought in= average number of sets in the system.

   = E(n) = e / (i- e) = 1.25/ (2-1.25) = 1.666
2. Time taken to repair one set = ½ hours

   Number of sets arriving per day =10

   Time taken to repair 10 sets = l0x *1/2 = 5* hours

   The idle time per day = 8-5=3 hours.

Problem-3: At a one-man barber shop customers arrive according to poisson distribution with a mean arrival rate of 5 per hour and his hair cutting time is exponentially distributed with an average haircut taking 10 minutes. It is assumed that because of his excellent reputation customers are willing to wait. Calculate the following :

1. Average number of customers in the shop and the average number of customers waiting for a haircut.
2. The percentage of customers who have to wait prior to getting into the barber's chair.

Solution:

Mean arrival Rate (e) = 5 per hour and

Mean service rate (i) =60 / 10 = 6 per hour

Average **number of customers in the shop =** E(n)

$= e/(i-e) \quad = 5/ (6-5) = 5$

Average number of customers waiting for a haircut = E(m).

$= e^2 / i \times (i- e) = 25/ 6(6-5) = 25/6 \quad = 4.166$

Probability that a customer has to' wait= Probability that there is atleast one customer in the system = e/i=5/6=0.833

The percentage of customers who have to wait prior to getting into barbers chair = (5/6)xl00 = 83.33%

Problem - 4: A milk plant distributes its products by trucks, loaded at the loading dock The company has found that sometimes the trucks have to wait in queue and the company losses money. Thus it is suggested that either to go in for a second loading dock or discount prices equivalent to waiting time. Average arrival time is 3 per hour and the average service rate is 4 per hour. The local transport company provides 40% of the total trucks. Calculate

1. The probability that a truck has to wait
2. The waiting time of a truck.
3. Expected waiting time for the transport company truck per day.

Solution:

Mean arrival Rate (e) - 3 per hour and

Mean service rate (i) = 4 per hour

1. Probability that a truck has to wait

   = (e / i)) = 0.75

2. Expected waiting time of a truck

   = 1 / (i- e) = 1 hour

3. Expected waiting time for the transport company truck per day.

   = (Number of trucks/day) x (% of Pvt. Company trucks) x (Expected waiting time/ truck)

   = (3 x 24)x (0.4) x (3 / 4 (4-3)) = 21.6 hours per day.

4.6 MULTIPLE CHANNEL QUEUING MODEL: (M/M/C) : (GD/∞/∞)

In a multiple- channel queuing system, two or more channels (or servers) are available to handle customers who arrive for service. It covers situations where, for example, there may be more than one runway at an airport for takeoff and landing, there may be more than one doctor in a hospital OPD whom the patients can visit, and there may be more than one teller in the bank.

The most common and basic multiple-channel system contains parallel stations serving a single queue on a 'first come first serve' basis. The service stations provide the same services. The single queue may separate into shorter queues in front of respective service stations. Also, when it is advantageous, calling units (customers) can shift from one queue to another.

ASSUMPTIONS OF THE MODEL

The Multiple-channel model is based on the following assumptions :

1. The input population is infinite.
2. A single waiting line is formed.
3. The service is on a first come first served basis.
4. The arrival of customers follows poisson probability law and service time has exponential distribution.
5. There are k service stations, each of which provides identical service.
6. The arrival rate is smaller than the combined service rate of all k service facilities.

NOTATIONS USED IN MULTIPLE CHANNEL MODEL

With the number of calling units represented by n and the number of channels or service stations by k, the descriptive characteristics of the multi channel system are listed below:

e = Average rate of arrivals (mean arrival rate)

i = Average service rate of each of channel (mean service rate)

k *i* = Mean combined service rate of all the channels.

$$P = \frac{\ddot{e}}{`I}$$

1) Probability that there shall exactly be n customers in the system

$$Pn = \frac{(i)^n}{n!}.\ Po \text{ when } n = k \text{ and}$$

$$Pn = \frac{(e/i)^n}{k!k^{n-k}}.\ Po \text{ when } n > k \text{ and}$$

2) The expected or average number of customers in waiting line

$$\frac{(e/i)^{k+1}}{(k-1)!(k-e/i)^2}$$

3) The expected number of customers in the system

$$Ls = Lq + \frac{e}{i}$$

4) The average time a customer spends in the queue waiting for service

$$Wq = \frac{Lq}{i}$$

5) The average time a unit spends in the waiting line or being serviced in the system

$$Ws = Wq + \frac{1}{i}$$

Problem -1: A company has two manufacturing shops and two tool cribs, one for each shop. Both the tool cribs handle almost identical tools, ganges and measuring instruments. Analysis of service time shows a negative exponential distribution with a mean distribution of 2.5 minutes per workman. Arrivals of workmen follow poisson distribution with a mean of 18 per hour. The production manager feels that if tool cribs are combined for both shops efficiency will improve and waiting time in the queue will reduce. Do you agree with his opinion?

Solution :

First we have two tool cribs, one for each shop, where

Mean arrival rate e =18/hours and

mean service rate = i =60 /2.5 =24 / hour

The expected waiting rime in queue is given by

Wq = e / i (i - e)

= 18/24(24-18) =7.5 minutes

Now if the tool cribs are combined for both shops, then Mean arrival rate ==-18+18 *=36* /hr and mean service rate =24/hr The expected waiting time in queue = Wq = Lq / i = 45/14 minutes.

Thus the waiting time in the queue has been reduced and so we agree with the production manager,

Problem-2: Bank customers can be served from any one of the three different services points. They are observed to enter the bank in a poisson fashion at an average rate of 24 per hour, while the average time a bank clerk takes to service a customer is considered to be exponentially distributed at an average time of 6 minutes.

As an alternative the bank is thinking of installing an automatic servicing machine which although it will mean only one service channel, it will be able to serve individual customers three times as the bank clerks do at present. Advice the bank on which of the system would be the more efficient in terms of customer waiting time

Solution:

Current system:

Mean arrival rate e =24 / hours and

Mean service rate = i =60 *16* =10 / hour

The expected waiting time in queue

Wq = Lq/i

Substitute the values in the above equation Wq =6.5 minutes

Proposed system

Mean arrival rate e =24 / hours and

Mean service rate = *130 /* hour

Average time in queue

Wq = e / i (i - e)

= 8 minutes Hence in terms of waiting time, the current multi-channel system is superior.

4.7 SUMMARY

In general the queuing system consists of one or more queues. Since the number of customers arriving to a service station can not be exactly predicted, it is very difficult to decide number of service stations or number of server require to provide the service. The queuing theory helps is determining this taking into consideration the various as parameters such as numbers of possible customers and number of customers that are allowed in a system and so on.

4.8 KEY WORDS

Poisson Distribution,

Single server,

Multiple server,

Finite customers,

Infinite customers

4.9 SELF ASSESSMENT QUESTIONS

1. The management of the Zen transport service company is concerned about the amount of time of the company trucks are idle, waiting to be unloaded. The terminal operates with four unloading bays. Each bay requires a crew of two employees; each crew costs Rs 50 per hour. The estimated cost of an idle truck is Rs.200 per hour. Trucks arrive at an average rate of three per hour according to Poisson distribution. A crew can unload a semi trailer rig in an average of one hour, with exponential service times. What is the total hourly cost of operating the system?

2: Apollo hospital provides free dental service to the patients on every morning. There are three dentists on duty, who are equally qualified and experienced. It takes, on an average 20 minutes for a patient to get treatment, and the actual time taken is known to vary approximately exponentially around this average. The patient arrives according to the Poisson distribution with an average of 6 per hour. The medical superintendent of the hospital wants to investigate the following :

1. The expected number of patients waiting in the queue
2. The average time that a patient spends at the clinic
3. The probability that a patient has to wait before he gets service
4. The average percentage idle time for each of the dentists.

3. In a Bank, every 15 minutes one customer arrives for cashing the cheque. The staff in the only payment counter takes 10 minutes for serving a customer takes 10 minutes for serving a customer on an average. State suitable assumption and find

   1. The average queue length.
   2. Increase in the arrival rate in order to justify a second counter (when the waiting time of a customer is at least 15 minutes the management will increase one more counter)

4. Southern Bank is considering opening a drive-in window for customer service. Management estimates that the customer will arrive for service at the rate of 15 per hour. The teller whom it is considering to staff the window can service the customer at the rate of le every 3 minutes. Assuming poisons arrival and exponential service time find

1. Utilization of the teller.
2. Average number in the waiting line
3. Average number in the system
4. Average waiting time in the line
5. Average waiting time in the system.

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UNIT 5 : DECISION ANALYSIS

STRUCTURE

5.0 Objectives

5.1 Introduction

5.2 Decision Making

5.3 Decision making under certainty

5.4 Decision making under uncertainty

5.5 Decision making under Risk

5.6 Decision making under Conflict

5.7 Decision Tree Analysis

5.8 Decision making Under Utility

5.9 Summary

5.10 Key Words

5.11 Self Assessment Questions

5.12 References

5.0 OBJECTIVES

After studying this unit, you should be able to :

* Explain role of decision analysis in real life situations;
* Discuss the importance of decision making;
* Examine different conditions under which decision are made and
* Construct decision trees

5.1 INTRODUCTION

Decision making is not a new function. In our daily life, we take up decisions over one or another occasion. These decisions may have a short term or long term effect in our lives.

Similarly the decision taken by the companies or their representatives would certainly affect the company in long run.

These decisions may include

- Acquiring of another company
- Making or buying of parts
- Buying or selling of shares
- Introduction of new product into market
- Fixing up of wages etc.,

5.2 DECISION MAKING

A decision can be defined as the selection by the decision maker of an act which he/she considers as best according to some pre-designated standard from among several available options.

Decision making is a process of selecting the best amongst the available alternatives.

Decision making problem

The Decision making problem includes the following element

1. Course of action

A decision is made from a set of different alternative course of actions or acts or strategies.

2. State or strategies

The state of nature refers to consequence of decision making which are beyond control.

3. Uncertainty

In some cases of decision making all the influencing factors may not be known to the decision maker. This is called as uncertainty.

4. Payoff:

The decision making now yields which are called as pay off. It would be gain or loss. The pay off under different consequences subjected alternative courses of actions can be listed in the form of a table which is called as pay-off table.

State of nature	**Alternative course of action**				
	A_1	A_2	A_3	A_j	A_n
C1	P_{11}	P_{12}	P13	P_{lj}	P_{ln}
C2	P_{21}	P_{22}	P_{23}	P_{2j}	P_{2n}
C3	P_{31}	P_{32}	P_{33}	P_{3j}	P_{3n}
Ci	P_{il}	P_{i2}	P_{i3}	P_{ij}	P_{in}
Cn	P_{ml}	P_{m2}	P_{m3}	P_{mj}	P_{mn}

Decision making process

There are different steps to be followed while taking a decision.

Step 1 : Determine the various alternative course of action..

Step 2 : Identify the possible outcomes

Step 3 : Determine the pay off function which describes the consequences resulting from decision alternative.

Step 4: Construct the opportunity loss table. The opportunity loss occurs due to failure of not adopting the best available course of action.

Consider a fixed state of nature $E_{;}$ (i = 1, 2 …..m) for which the pay off corresponding to the n course of action given be P_{11} P_{12}.....Pm.

Let MI be the pay off of the least possible course of action. The opportunity cost table is created as below.

State of nature	Conditional Opportunity loss				
	A_1	A_2	A_3	Aj	A_n
C_1	M_rP_{11}	M_rPi_2	M_1P_{13}	M_1Pij	M_1P_{fa}
C_2	M_2-P_{12}	M_2-P_{22}	M_2-P_{23}	M_2-P_{2j}	M_2-P_{2n}
C_3	M_3-P_{13}	M_3-P_{32}	M_3-P_{33}	M_3-P_{3j}	M_3-P_{3n}
C_i	M_1P_{li}	Mi-P_{i2}	MrPi3	MrPij	M_4-P_{in}
C_m	M_m-P_m	M_{ni}-P_{m2}	M_m-P_{mi}	M,-Pmj	Mn- Pim

Decision making Environment

There are mainly 4 situations under which decision are taken

a. ***Decision under certainty : -***

Here all the variables that supports the decision s are known to the decision maker. Hence the decision maker can come out with only one decision which is the best.

For eg. : Had we know the transportations cost from each source to each destinations & supply & demand of each source and Destination, Then we can decide how much to transport from which source to which destination.

Similarly if we know cost of manufacturing an item with in the plant & cost of procuring the same item from outside, then we can decide whether make or buy the parts.

b. Decision under uncertainty : -

Here the all factors that would have an effect on decision are not known to the decision maker. So decision making becomes difficult.

Decision making under uncertainty is like shooting at enemy at dark without knowing whether enemy is there or not.

Another example that we can give for decision making under uncertainty is telecasting of advertisements at different time without knowing exactly at what time prospective customer would watch the television.

c. ***Decision making another risk.***

This refers to a situation where decision maker chooses from several possible out comes where in probabilities of occurrences can be stated.

The decision taking under risk is like shooting a enemy at dark by knowing that enemy is there.

Decision making under risk is involved launching a new product even after preliminary survey made.

d. ***Decision Under Conflict***

In many situations neither states of nature are completely known nor are they completely uncertain. Partial knowledge is available and therefore it may be termed as decision making under partial uncertainty. An example if this is the situation of conflict involving two or more competitors marketing the same product.

Probabilities may be based on decision maker's personal opinions about future events, or on data obtained from market surveys, expert opinions, etc.

When probability of occurrence of each state of nature can be assessed, problem environment is called decision making under risk.

Examples: -

- Probability of being dealt club from deck of cards is 1/4.
- Probability of rolling 5 on die is 1/6.

5.3 DECISION MAKING UNDER CERTAINTY

Certainty is when you have information about the event, you know the outcome of the event and you are confident of the events happening. We experience certainty about a specific question when we have a feeling of complete belief or complete confidence in a single answer to the question.

Decisions such as deciding on a new carpet for the office or installing a new piece of equipment or promoting an employee to a supervisory position are made with a high level of certainty.

While there is always some degree of uncertainty about the eventual outcome of such decisions there is enough clarity about the problem, the situation and the alternatives to consider the conditions to be certain.

It is the term used in a situation where for each decision alternative there is only one event and therefore only one outcome for each action. For example, there is only one possible event for the two possible actions: "Do nothing" at a future cost of S3.00 per unit for 10,000 units, or "rearrange" a facility at a future cost of $2.80 for the same number of units. A decision matrix (or payoff table) would look as follows:

Actions	State of Nature (with probability of 1.0)
Do nothing	$30,000 (10,000 Units.$3.00)
Rearrange	28,000 (10,000 units $2.80)

Note that there is only one State of Nature in the matrix because there is only one possible outcome for each action (with certainty). The decision is obviously to choose the action that will result in the most desirable outcome (least cost), that is to "rearrange.".

5.4 DECISION MAKING UNDER UNCERTAINTY

Under this condition the probabilities associated with occurrence of different status of nature are not given but the pay off are given.

Different approaches are used here to determine the solution.

Laplace criterion

This criterion assigns equal probability to all possible pay off and then select the alternative which corresponds to the maximum pay off.

For eg. : Suppose you want to buy lottery ticket worth Rupees 10, 3 different tickets are available having reward of 10,000, 1,00,000 and 10,00,000. Then you would buy the last one.

Problem :

A super Bazaar expects the sales in one of the four categories 300, 350, 400 or 450. The cost is as given deviation from ideal level results in additional cost of stocking cost or demand not full fill.

Customer Category	Inventory level			
	A1	A2	A3	A4
E_i -300	7	12	20	27

E_2-350	10	9	10	25
E_3 - 400	23	20	14	23
E_4 - 450	32	24	21	17

Laplace criteria assumes equal probabilities i.e., 1/4 for each

The net pay off includes

$E(A_1)$=l/4(7 + 10 + 23+32)=18

$E (A_2) = 1/4 (12 + 9 + 20 + 24) = 16.25$

$E (A_3) = 1/4 (20 + 10 + 14 + 21) = 16.25$

E (A4) = 1/4 (27 + 25 + 23 + 17) = 23

The best of level inventory is either A_2 or A_3 which is minimum

The maximin or mini max criterion

It assumes that worst possible is going to happen. Select that which maximizes the minimum pay off

Say for example

There are 3 schemes of mutual funds, cost being same.

Scheme	Guaranteed Income	Expected Income
Scheme	1000	100000
Scheme 2	1200	50000
Scheme 3	1100	75000

Then according this principle scheme 2 is selected.

The method followed is as below

Step 1: Determine minimum pay off of each alternative

Step 2 : Choose maximum out of this

It can be (maxi min) maximum of minimum for profit.

Or (Mini max) minimum of maximum for cost.

	A1	A2	A3	A4
E1	7	12	20	27
E2	10	9	10	25
E3	23	20	14	27
E4	32	24	21	17

here the maxi mum cost of each all is

A_1-32, A_2-24, A_3-21, A_4-27

Minimum of this is 21 i.e., A_3 T

Therefore A3 is selected,

The maxi max or minimin categories

This is based on optimism Here the decision maker selects the maximum of maximum pay off or min of min of cost Taking the same example.

	A_1	A_2	A_3	A_4
E_1	7	12	20	27
E_2	10	9	10	25
E_3	23	20	14	23
E_4	32	24	21	17

Here the min cost is

A_1-7, A_2-9, A_3-10, A_4-17

Minimum of all there is 7

So A_1 is selected

17.4.4 Savage Criteria

This is based on regret or opportunity and calls for selecting the course of action that minimize the maximum regret.

Step -1 : Determine the amount of regret for pay off of each alternative for a particular event which is calculated as below.

i^m regret = (max payoff- i th pay off) for i^{th} event if he pay off represents profit

(i^{th} pay off - min. pay off) for i^{th} event if the pay off represents cost

Step 2 : Determine max regret for each alternative

Step 3 : Select minimum out of these

Considering the same example.

	A 1	A 1	A 1	A 1	Min pay off
E 1	7	12	20	27	7
E 2	10	9	10	25	9
E 3	23	20	14	23	14
E 4	32	24	21	17	17

Deduct this minimum pay off from all the elements of that row.

Regret payoff amount

	0	5	13	20
	1	0	1	16
	9	6	0	9
	15	7	4	0
Max regret	15	7	13	20

Minimum of this is 7

So A_2 is chosen

17.4.5 Hurwicz Criterion

It stipulates that a decision maker's should be both optimist and pessimistic.

Steps

1. Choose a as degree of optimism and 1 - a as degree of pessimism
2. Determine. Max & min pay off for each alternative
3. Calculate n=a xI + (l-a)II

Consider the same example

	E_1	E_2	E_3	E_4	I max	II min
A_1	7	10	22	32	32	7
A_2	12	9	20	24	24	9
A_3	20	10	14	21	21	10
A_4	27	25	23	17	27	17

Let us take α= 0.5

N= α x I + (l-α)x II

A_1-» (0.5X32 + (1-.5) X 7) =19.5

A_2-» 16.5

A_3-» 15.5

A_4-»22

So chose A3 as it is minimum

5.5 DECISION MAKING UNDER RISK

When a decision maker chooses from several possible alternatives whose probability of occurrence is stated then the decision taken is called decision under risk.

Here EMV or Expected Monetary Value as calculated.

The expected monetary value for a given course of action is the expected value of the additional pay off for that action.

Step 1 : List conditional profit for each act / event

Step 2 : Determine the expected conditional profit

Step 3 : Determine EMV for each act

Step 4 : Choose the act that corresponds to optimal EMV

Problem :

A man has choice of running either a hot snack stall or ice- cream stall at a sea side resort. If it is fairy cool summer he should make Rs. 5000 by running the hot snack stall. If it is not he can make profit of Rs. 1000. On the other hand his profit is Rs. 6,500 for hot summer and Rs. 1000 if it is cool by running ice cream stall. There is 40% chance of summer being hot. What should be his choice.

Solution

The pay off table can be constructed as below.:

Conditional pay off

Event	Probability Ei	Hot snack	Ice -cream
Cool summer	0.6	5000	1000
Hot summer	0.4	1000	6500

Expected conditional pay off

i	ii	Iii	iv
Event	Prob	Hot (i x ii)	Cool (I x iii)
Cool	0.6	3000	600
Hot	0.4	400	2600
		3400	3200

Since the expected monetary value of selling hot snack is more he should opt for hot snack.

5.6 DECISION MAKING UNDER CONFLICT

Conflict and choice are closely related in that choice produces conflict and conflict is resolved by making a choice. Although conflict was invoked in psychological approaches to decision making , no generally accepted measure of conflict strength has been established.

Here the interest of two varied personnel comes into picture.

In decision making under conflict many outcomes are possible. An enemy is trying to outwit you & foil your strategy.

Usually these types of problems are solved using Game theory which are going to be discussed next.

5.7 DECISION TREE ANALYSIS

It is graphic display of various decision alternatives and sequence as if they were branches of a tree.

- Decision point
- Event

Problem

Amar company is currently working with a process which after paying for materials labour, etc. brings a profit of Rs. 12,000 . The following alternatives are made available to the company.

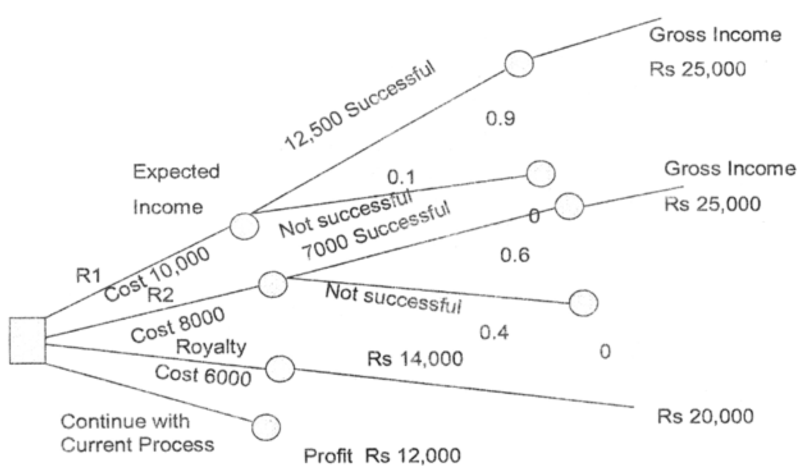
(i) The company can conduct research (R_1) which is expected to cost Rs. 10,000 having 90% chances of success. If it proves a success the company gets a gross income of Rs. 25,000

(ii) The company can conduct research (R2) which is expected to cost Rs. 8000 having probability of 60% success. The gross income will be 25,000

(iii) The company can pay Rs. 6000 as royality for a new process which will bring a gross income of Rs. 20,000.

(iv) The company can continue the current process

Draw the decision tree as follows



Now these values are tabulated in the table below

Decision Analysis

Decision	Even	Probability	Income	Expected Income Probability Income
1. conduct Research not R_1	Successful successful	0.9 0.1	25,000 0	22,500 0
			Total Expected Income	22,500
			Less cost	10,000
				12,500
2. conduct Research not R_2	Successful successful	0.6 0.4	25,000 0	15,000 0
				22,500
				-8,000
				7,000
3. Pay Royalty	Certain	1	20,000	20,000-6000=14000
4. Continue same process	Certain	1	12,000 .	12,000

The net EMV is highest for the alternative pay off royalty for the new process, the optimal decision would be the procure a process on Royalty basis.

5.8 DECISION MAKING UNDER UTILITY

Always money can not be the sole criteria. In may case decisions has to be taken. Also even when expected values are calculated in money terms, money may mean every thing. Some people would prefer to take risk, some would take prestige in starting new venture.

A rational decision maker will choose that alternative which optimise the expected utility rather than expected monitory value. Once we know the individual utility function along with the probability assigned to out come in a particular situations then total expected utility for each course of action can be obtained by multiplying utility values with their probability.

5.9 SUMMARY

Decision making is an integral part of most planning organizing controlling & motivate process. The decision maker selects one strategies or course or action over other depending upon utility sales, cost or rate of return. Decision theory provides a method for rational decision making when the consequences are not fully deterministic. They provide a frame work for better understanding of the decision situation for evaluating alternatives.

5.10 KEY WORDS

* Decision making under Certanites
* Decision making under Uncertainity
* Decision making under Risk

5.11 SELF ASSESSMENT QUESTIONS

1. Define decision making. Explain decision making under various circumstances.
2. A food company is thinking of launching a new product at high price (S_1) or moderate change in existing product with small increase in price (S_2) or small change in composition with negligible price change (S_3) as a result of which
3. There can be increase in sales (I)
4. No change in sales (N)
5. Decrease in sales (D)

Which would have an input on profit as show.

State of Nature	S_1	S_2	S_3
I	7,00,000	5,00,000	3,00,000
N	3,00,000	4,50,000	3,00,000
D	1,50,000	0	3,00,000

Solve this problem using all the different criterion

4. A farmer is attempting to decide which of three crops he should plant on his one hundred acre farm. The profit from each crop strongly depend on rainfall. He categorize the amount of rainfall as substantial, moderate or light. His profit is estimated below.

Rainfall	Crop A	Crop B	Crop C
S	7000	2500	4000
M	3500	3500	4000
L	1000	4000	3000

The estimated probability of substantial rainfall is 0.2, moderate is 0.4 & light is 0.5.

5. A glass factory specialized in crystal is developing a substantial backdrops in farms management considering 3 course of action.

SI arrange for sub contract, S2 arrange over time S3 construct new facility. The future demand may be low, medium or high the probability which is 0.1, 0.5 & 0.4

The profit matrix is as shown.

Profit	S_1	S_2	S_3
Low	10	-20	-150
Medium	50	60	20
High	50	100	200

Construct a decision and indicate preferred action.

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UNIT 6 : NETWORK ANALYSIS

STRUCTURE

6.0 Objectives

6.1 Introduction

6.2 Network Analysis

6.3 Application of PERT and CPM techniques to Business problems.

6.4 Distinction between PERT and CPM.

6.5 Basic concepts of Network Analysis

6.6 Rules of Network Construction

6.7 Fulkerson's Rule (i-j Rule) of Numbering Events

6.8 Illustrations

6.9 Summary

6.10 Key Words

6.11 Self Assessment Questions

6.12 References

6.0 OBJECTIVES

After studying this unit, you should be able to:

* Explain the importance of network analysis;
* Examine the various applications of network analysis;
* Distinguish between P E R T and C P M;
* Explain the basic concepts of network analysis;
* Learn the rules of network analysis;
* Get acquainted with Fulkerson's Rule of computing and
* Construct the network of simple problems.

6.1 INTRODUCTION

A project such as construction of a bridge, highway, flyover, power plant, repair and maintenance of oil refineries or an air plane; design, development and marketing of a new product; research and development work, etc. may be defined as a collection of interrelated activities (tasks) which must be completed in a specified time according to a specified sequence (or order) and require resources such as personnel, money, materials, facilities. The process of dividing the project into these activities is called the work breakdown structure (WBS). The activity or a unit of work, also called work content, is an identifiable and manageable work unit. The main objective before starting such projects is: How to schedule the required activities so as to:

a. Complete the given project on or before a specified time limit.
b. Minimize the cost of completion of the project on or before a specified time limit.
c. Minimize the total project completion time for a given cost.

Hence, before starting any project, it is essential to devise an adequate plan for scheduling and controlling the various activities (tasks) of the given project. The class of operations research techniques used for planning, scheduling and controlling large and complex projects are often referred to as network analysis, network planning, or network planning and scheduling techniques. PERT and CPM are two well known techniques used for network analysis.

6.2 NETWORK ANALYSIS

A network is a graphical diagram consisting of a certain configuration of arrows and nodes for showing the logical sequence of various activities (tasks) to be performed to achieve project objectives. Network analysis is quite useful for designing, planning, co-coordinating, controlling and decision making so that the project could be economically completed in the minimum possible time with the limited available resources. The two most popular form of this technique now used in many scheduling situations are the Programme Evaluation and Review Technique (PERT) and the Critical Path Method (CPM).

Programme Evaluation and Review Technique (PERT)

PERT was developed in 1956-58 by the US Navy Special Projects office in co-operation with the management consulting firm of Booz, Alien and Hamilton to aid in the planning and scheduling of the US Navy's Polaris Missile Programme which involved over three thousand different contracting organizations. The Objective of the team was to efficiently plan and produce the Polaris missile system. Since then this technique has proved to be useful for all jobs or projects which have an element of uncertainty in the matter of estimation of duration, as in case with new types of projects both at the Government and Industry level. In PERT we usually assume that the time to perform each activity is uncertain and as such three time estimates i.e. is the optimistic, the pessimistic and the most likely time estimates are used.

Critical Path Method (CPM)

CPM was developed in 1957 by J.E.Kelly of Remington Rand and M.R. Walker of E.I.Dupont to aid in the scheduling of routine plant overhaul, maintenance and construction of work. This method differentiates between planning and scheduling .Planning refers to the determination of activities that must be accomplished and the order in which such activities should be performed to achieve the objectives of the project. Scheduling refers to the introduction of time into the plan thereby creating a time table for the various activities to be performed. CPM uses two time and two cosl; estimates for each activity. CPM operates on the assumption that the time taken by each activity in the project is already known precisely.

6.3 APPLICATION OF NETWORK ANALYSIS TO BUSINESS

Few Management applications of PERT and CPM are to plan, schedule, monitor and control projects such as;

i. Construction of buildings, bridges, factories, highways, stadiums, irrigation projects, etc.
ii. Budget and auditing procedures.
iii. Missile development programmes.
iv. Installation of complex new equipments such as computers or large machinery.
v. Advertising programmes and for development and launching of new products.
vi. Planning of political campaigns.
vii. Strategic and tactical military planning.
viii. Research and development of new products.
ix. Finding the best traffic flow pattern in large cities.
x. Maintenance and overhauling complicated equipments in the chemical, power plants steel and petroleum industries.
xi. Long range planning and developing staffing plans.
xii. Organising of big conferences, public works, etc.
xiii. Shifting of manufacturing plant from one site to another.
xiv. Preparation of bids and proposals for projects of large size.
xv. Launching space programmes.

6.4 DISTINCTION BETWEEN P E R T AND C P M

The basic differences between the two techniques are summarized below;

P E R T

1. A probability model with uncertainty in activity duration. The duration of each activity is normally computed from multiple time estimates with a view to take into account time uncertainty. These estimates are ultimately used to arrive at the probability of achieving any given scheduled data of project completion.

2. It is said to be event oriented network because in the analysis of network emphasis is given on important stages of completion of tasks rather than the activities required to

3. PERT is normally used for projects involving activities of non-repetitive nature in which time estimates are uncertain.

4. It helps in pin pointing critical areas in a project so that necessary adjustments can be made to meet the scheduled completion date of the project.

5. PERT analysis does not usually consider costs.

C P M

1. A deterministic model with well known activity times based upon past experience. It, therefore, does not deal with uncertainty in time.

2. CPM is suitable for establishing a trade-off for optimum balancing between scheduled time and cost of the project.

3. CPM is used for projects involving activities of repetitive nature.

4. CPM deals with costs of project schedules and their minimization. The concept of crashing is applied mainly to CPM models.

5. It is difficult to use CPM as a controlling device for the simple reason that one must repeat the entire evaluation of the project each time; the changes are introduced into the network.

6.5 BASIC CONCEPTS OF NETWORK ANALYSIS

A fundamental ingredient in both PERT and CPM is the use of network systems as a means of graphically depicting the current problems or proposed project. PERT and CPM network consists of two major components as discussed below:

Event

Events of the network represent project milestones, such as the start or the completion of an activity (task) or activities, and occur at a particular instant of time at which some specific part of the project has been or is to be achieved; therefore events do not consume time or resources. Events are commonly represented by circles. The event circles are called nodes in the network diagram. Events can be further classified into following types:

Merge Event

When more than one activity comes and joins, the event is known as merge event.

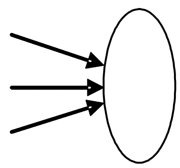

Figure 3.1 Merge Event

Burst Event

When more than one activity leaves an event, the event is known as a burst event.

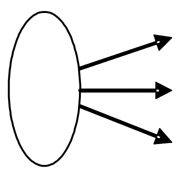

Figure 3.2 Burst Event

Merge and Burst Event

An activity may be a merge and burst event simultaneously as with respect to some activities it can be merge event and with respect to some other activity it may be burst event.

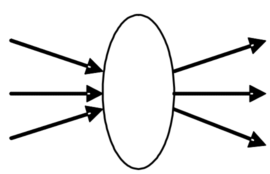

Figure 3.3 Merge and Burst Event

Activity

Activities of the network represent project operations or tasks to be conducted. Activities require the expenditure of time and resources for their accomplishment. Dummy activities do not consume time and resources. An arrow is commonly used to represent an activity with its head indicating the direction of the progress in the project. The activity arrow is called arc. The activity arrow is not scaled; the length of the arrow is a matter of

Activities are identified by the number of their starting (tail) event and ending (head) event. An arrow (i, j) extended between two events, the tail event T represents the start of the activity and the head event 'j', represents the completion of the activity.

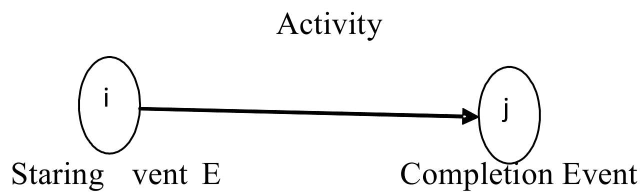

Figure 3.4 Activity Node Relationships

The activities are further classified into following three types:

Predecessor Activity

An activity which must be completed before one or more other activities start is known as predecessor activity.

Successor Activity

An activity which started immediately after one or more of other activities are completed is known as successor activity.

Concurrent activity

Activities which can be accomplished concurrently/simultaneously are known as concurrent activities. It can be observed that an activity can be a predecessor or successor to an event may be concurrent with one or more of the other activities.

Dummy Activity

An activity which does not consume either any resource or time is known as dummy activity. A dummy activity in the network is added only to represent the given. A dummy activity is depicted by dotted lines in the network diagram, precedence relationships among activities of the project and is needed when,

1. Two or more parallel activities in a project have same head and tail events or

2. Two or more activities have some (but not all) of their immediate predecessor activities

in common.

Sequencing

The first prerequisite in the development of a network is to maintain the precedence relationships. In order to construct the network following points have to be taken note of;

- What job or jobs precede an activity?
- What job or jobs could run concurrently?
- What job or jobs follow it?
- What controls the start and finish of a job?

6.6 RULES OF NETWORK CONSTRUCTION

There are a number of rules in connection with the handling of events and activities of a project network that should be followed:

1. Try to avoid arrows which cross each other.
2. Use straight arrows.
3. No event can occur until every activity preceding it has been completed.
4. An event cannot occur twice.
5. An activity succeeding an event cannot be started until that event has occurred.
6. Use the arrows from left to right. Avoid mixing two directions; vertical and standing arrows may be used if necessary.
7. Dummies should be introduced if it is extremely necessary.
8. The network has only one entry point called the start event and one point of emergence called the end or terminal event.

6.7 FULKERSON'S RULE (i-j rule) NUMBERING EVENTS

After the network is drawn in a logical sequence every event is assigned a number. The number sequence must be such so as to reflect the flow of the network. The following rules are followed when numbering the events.

1. Number the start node which has no predecessor activity as 1.
2. Delete all the activities emerging from this node 1.
3. Number all the resulting start nodes without any predecessor as 2, 3...
4. Delete all the activities originating from the start nodes without any predecessor next to the last number in step3.
5. Number all the resulting new start nodes without any predecessor next to the last number in step3.

6.8 ILLUSTRATIONS

After learning the concepts and rule of constructing the networks, let us try to solve some simple problems.

Problem-1:

Draw a network for the following project and number the events according to Fulkerson's rule.

1. A is the start activity and K is the end activity.
2. J is the successor activity to F.
3. C and D are successor activity to B.
4. D is the preceding activity to G.
5. E and F occur after event C.
6. E precedes J.
7. Restrains the occurrence of J and G precedes H.
8. K succeeds activity.

Solution:

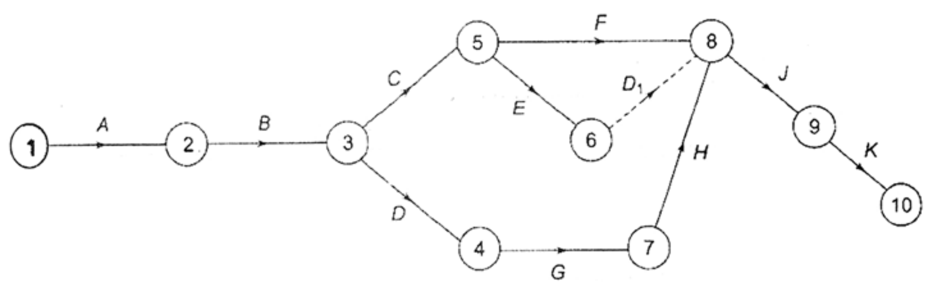

Figure 3.5

Problem-2:

Draw a network for the following data.

Event No.	1	2	3	4	5	6	7
Immediate Predecessors	-	1	1	2,3	3	4,5	5,6

Solution

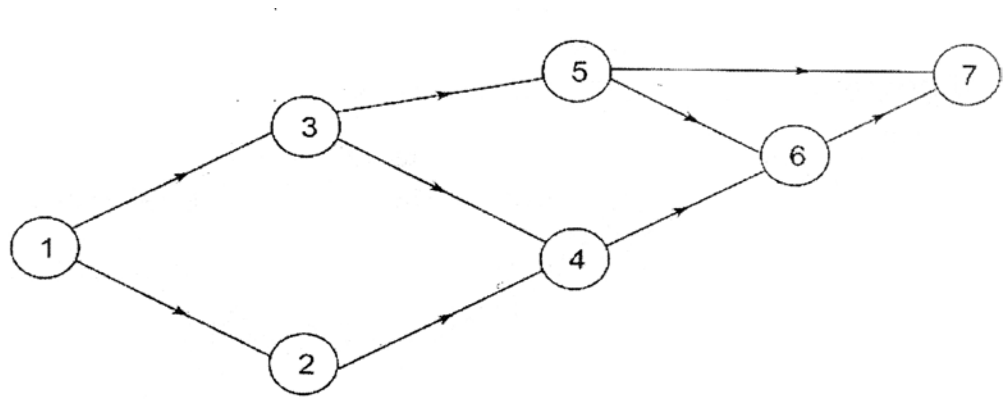

Figure 3.6

Problem- 3:

Construct a network for the project whose activities and their precedence relationship are given below:

Activity No	A	B	C	D	E	F	G	H	I
Immediate Predecessors	-	A	A	-	D	B,C,E	F	D	G,H

Solution

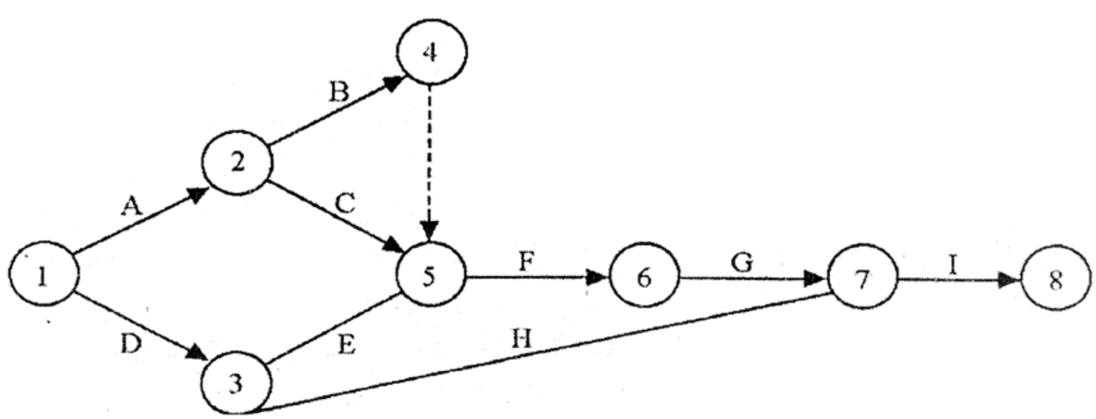

Figure 3.7

Problem- 4:

Draw the network using the data given below.

A < C, D, I; B < G, F ; D < G, F ; F < H, K ; G, H < J ; I, J, K < E

Solution:

Given A < C which means that C cannot be started until A is completed i.e. A is the preceding activity to C. The above constraints can be given in the following table.

Event No	A	B	C	D	E	F	G	H	I	J	K
Predecessors	-	-	A	A	I,J,K	B,D	B,D	F	A	G,H	F

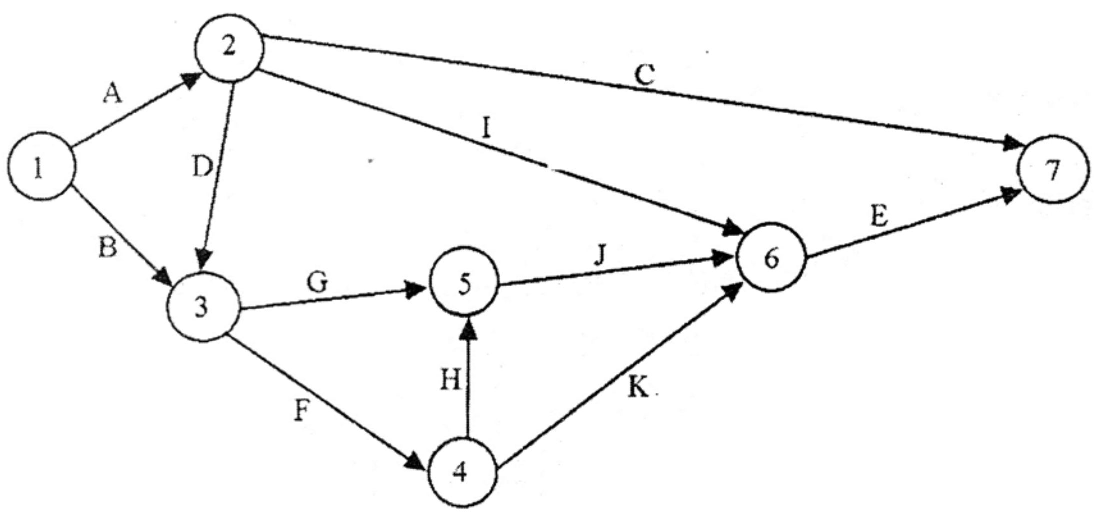

Figure 3.8

Problem-5:

The sequence of activities together with their predecessors for manufacturing an item are given below, draw the network diagram.

Activity	A	B	C	D	E	F	G	H
Predecessor Activity	-	A	A	B	B,C	E	D,F	G

Solution

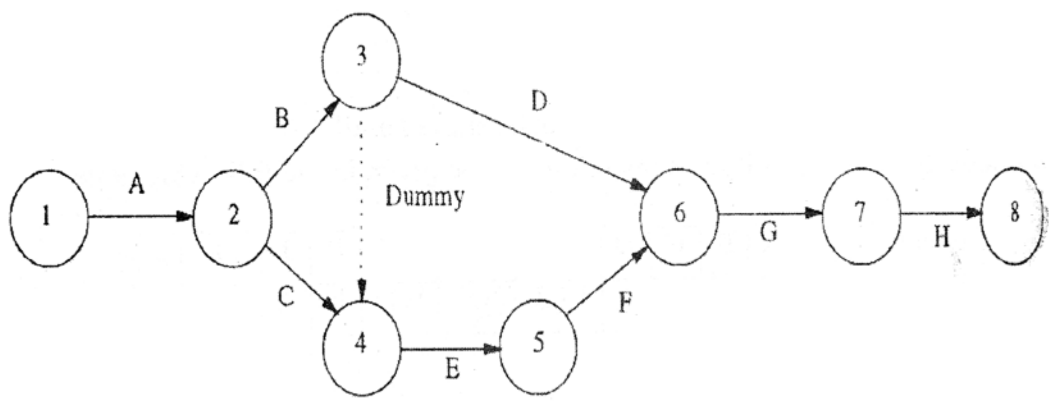

Figure 3.9

Problem-6:

Given in the table are the activities and sequence necessary for maintenance job of material handling equipments in a factory. Draw a network.

Activity	A	B	C	D	E	F	G	H	I	J
Predecessor Activity	-	A	B	B	B	C	C	F,G	D,E,H	I

Solution:

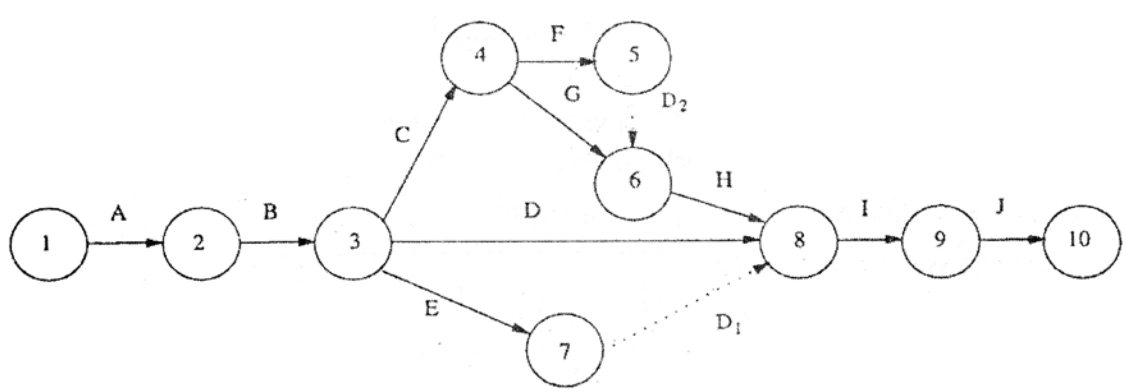

Figure 3.10

Problem - 7:

Activity	A	B	C	D	E	F	G	H	I
Pre-requisite Activity	-	-	-	A	B	C	D,E	B	D,E,H

Solution

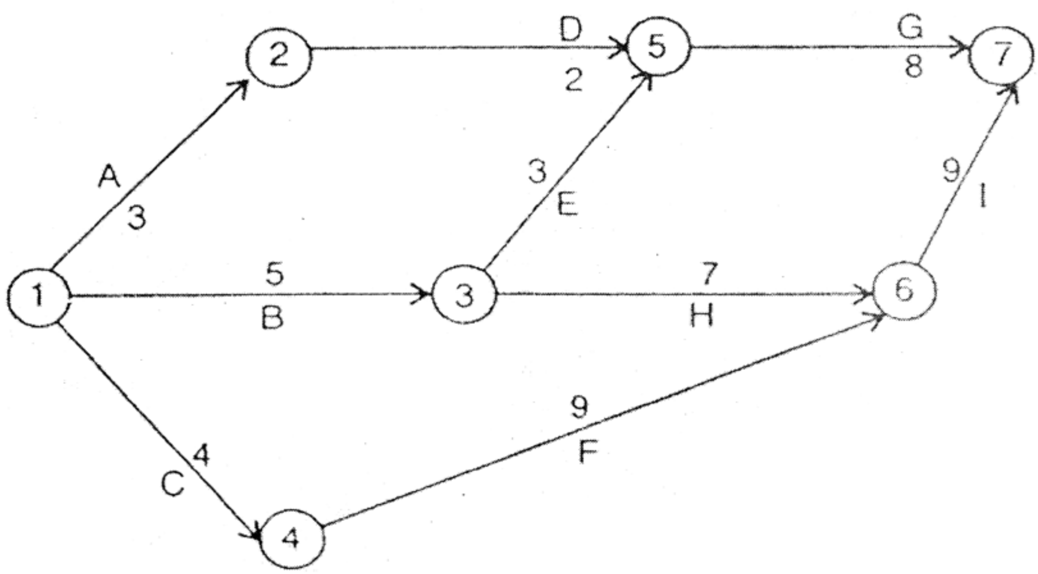

Figure 3.11

6.9 SUMMARY

Network analysis helps in monitoring a project to ensure must be project would be completed within

6.10 KEY WORDS

Activity

Event

Merge event

Burst event

Predecessor

Successor

Sequencing

Network

6.11 SELF ASSESSMENT QUESTIONS

Case Study-1:

Sigma Ltd., an Airplane manufacturing company wants to draw a network for their project. The following information is available. Draw the project for Sigma Ltd.

Activity	A	B	C	D	E	F	G	H	I	J	K
Pre-requisite Activity	-	A	A	C	B,C	D,E	E	G	D,F	I,H	J

Case Study - 2:

Zen Limited has a listed the activities and sequence requirements necessary for the machine maintenance in their company. Draw a network diagram.

Activity	A	B	C	D	E	F	G	H	I	J	K	L	M	N	0	P	Q	R	S	T
Pre-	-	-	B	A	C	B	A ,F	G	A ,F	G	J ,H	J	L	M ,K	N	O	P	E	Q ,D ,I ,R	R
requisite																				

6.12 REFERENCES

1. Gupta S.P. *Business Statistics*, New Delhi: S Chand and Sons Publishers, 2000
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UNIT 7 : SOLUTION TO NETWORK PROBLEMS

STRUCTURE

7.0 Objectives

7.1 Introduction

7.2 Notations used for basic Scheduling Computation

7.3 Forward Pass Method

7.4 Backward Pass Method

7.5 Critical Path

7.6 Determination of Float and Slack Time

7.7 Summary

7.8 Key Words

7.9 Self Assessment Questions

7.10 References

7.0 OBJECTIVES

After studying this unit, you should be able to ;

- Explain the objectives of critical path analysis.
- Compute the three time estimates.
- Prepare the project scheduling with the help of networks.
- Discuss forward and backward pass methods.
- Determine the float and slack time.

7.1 INTRODUCTION

The objective of critical path analysis is to estimate the total project duration and to assign staring and finishing times to all the activities involved in the project. This helps in checking actual progress against the scheduled duration of the project.

The duration of individual activities may be uniquely determined (in case of CPM) or may involve the three-time estimates (in case of PERT) out of which the expected duration of an activity is computed. Having computed this, the following factors should be known to prepare project scheduling:

1. Total completion time of the project.
2. Earliest and latest start time of each activity.
3. Float of each activity, i.e. the amount of time by which the completion of an activity can be delayed without delaying the total project completion time.
4. Critical activities and the critical path.

7.2 NOTATIONS USED FOR BASIC SCHEDULING COMPUTATION

The following notations are used for the basic scheduling computation:

1. T_E or E_i = Earliest occurrence time of event, i. It is the earliest time at which an even: can occur without affecting the total project time.
2. T_L or L_i = Latest occurrence time of event i. It is the latest time at which an event can occur without affecting the total project time.
3. ES_{ij} = Earliest start time for activity (i,j). It is the earliest time at which the activity can start without affecting the total project time.
4. LS_{ij} = Latest start time for activity (i,j). It is the latest possible by which an activity must start without affecting the total project time. \
5. EF_{ij} = Earliest finish time for activity (i,j). It is the earliest possible time at which a

activity can finish without affecting the total project time.

6. LF_{ij} = Latest finish time for activity (i,j). It is the latest time by which an activity mi get completed without delaying project completion.
7. t_{ij}= Duration of activity (i,j).

For calculating the above mentioned times, two methods namely forward pass and backward pass are employed.

7.3 FORWARD PASS METHOD

This method is used to compute earliest start time. In this method calculations begin from the initial event 1, proceed through the network visiting events in an increasing order of event number and end to the final event, At each event we calculate earliest occurrence event time and earliest start and finish time for each activity that begins at that event. When calculations end at the final event, its earliest occurrence time gives the earliest possible completion time of the entire project,

1. Set the earliest occurrence time of initial event to zero i.e. E_i= 0.
2. Calculate earliest start time for each activity that begins at event i (=1), This is equal to the earliest occurrence time of event, j (tail event) that is, $ES_{ij} = E_i$
3. Calculate the earliest finish time of each activity that begins at event i. This is equal to the earliest start time of the activity plus the duration of the activity. That is $EF_{ij} = ES_{ij} + t_{ij} = E_i + t_{ij}$ for all activities (i,j) beginning at event i
4. Proceed to the next event, say j ;j>i
5. Calculate the earliest occurrence time for event j. This is the maximum of the earliest finish times of all activities ending into that event, that is

   E_j = Max (EF_{ij}) = Max (E_i +1_{ij}) for all immediate predecessor activities.
6. If j =N, that is the final event, then the earliest finish time for the project, that is the earliest occurrence time E_N for the final event is given by,

   EN = Max (EF_{ij}) for all terminal activities

   = Max (E_{N-1} +1_{ij}) and stop the method.

7.4 BACKWARD PASS METHOD

This method is used to compute latest allowable time. In this method calculations begin from the final event N, proceed through the network visiting events in the decreasing order of event number and end at the initial event 1. At each event we calculate the latest occurrence time (L) for the corresponding event, the latest finish sand start for each activity that is terminating t the event, such that the earliest finish time for the project remains the same. The method may be summarized as follows:

1. Set the latest occurrence of the last event N equal to its earliest occurrence time (known from the forward pass method) that is, $L_N = E_N$ j=N.

2. Calculate the latest finish time of each activity which ends at event j . This is equal to the latest occurrence time of the final event N that is $LF_{ij} = L_j$ for all activities (i,j) ending at event j.

3. Calculate the latest start time of all activities ending at j . It is obtained by subtracting the duration of the activity from the latest finish time of the activity, that is
   $LF_{ij} = L$
   And $LS_{ij} = LF_{ij} - t_{i,j}$
   $= L_j - t_{ij}$ for ach activity (i,j) ending at event ,j.

4. Proceed backward to the event in the sequence that decreases j by t

5. Calculate the latest occurrence time of event i(i<j). This is the minimum of the latest start time of all activities starting from that event, that is Li =Min (LS_{ij}) for all immediate successor activities = Min (Lj - t_{ij})

6. If j=l (initial event), then the latest finish time for project, i.e. latest occurrence time L_j for the initial event is given by,

   L_1 = min (LS_{ij}) for all immediate successor activities.

   = min (L_j-t_{ij})

   and stop the method.

7.5 CRITICAL PATH

Certain activities in a network diagram of a project are called critical activities because delay in their execution will cause further delay in the project completion time Thus all activities having zero total float values are identified as critical activities.

The critical path is the continuous chain of critical activities in a network. It is the longest path starting from first to the last event and is shown by the thick line or double lines in the network diagram.

The length of the critical path is the sum of the individual times of all the critical activities lying on it and defines the minimum time required to complete the project.

The critical path on a network diagram can be identified as :

1. For all activities (i,j) lying on the critical path the E values and the L values for tail and head event are equal, that is $E_j=L_j$ and $E_j=L_j$.

2. On critical path E_j-E_l = L_j-L_i = t_{ij}

7.6 DETERMINATION OF FLOAT AND SLACK TIME

Float is defined as the difference between the latest and the earliest activity time. Slack is defined as the difference between the latest and earliest event time. There are three types of floats:

Total Float : It refers to the amount of time by which the completion of an activity could be delayed beyond the earliest expected completion time without affecting the overall project duration time.

Mathematically, the total float of an activity (i,j) is the difference the latest start time and the earliest start time of that activity.

Total float (TF_{ij}) = LS_{ij} - Es_{ij}

= (L_j-E_i) – t_{ij}

Free Float : The time by , which the completion of an activity can be delayed beyond the earliest finish time without affecting the earliest start time of a succeeding activity. Free float is calculated as Free float (FF_{ij}) = (E_j-E_i)-t_{ij}

Independent Float : The amount of time by which start of an activity can be delayed without affecting the earliest start time of any immediately following activity assuming that the preceding activity has finished at its latest finish time. The negative value of independent float is considered as zero. Independent float is calculated as, Independent float (IF_{ij}) = (E_j - L_i)-t..

= (ES_{ij}-LS_{ij}) - t_{ij}

Problem - 1: Kaizen Limited has decided to add - new product to its line, It will outsource the product from another firm, package it, and sell it to a number of distributors. The following are the activities to be completed to implement the above project.

Activity	Description	Time (weeks)
A	Organize sales office	6
B	Hire Salesmen	4
C	Train Salesmen	7
D	Select advertising agency	2
E	Plan advertising campaign	4
F	Conduct advertising campaign	10
G -	Design package	2
H	Setup packaging facilities	10
I	Package initial stocks	6
J	Order stock from manufacturer	13
K	Select distributors	9
L	Sell to distributors	3
M	Ship stock	5

The precedence relation between each activity is shown in the following diagram;

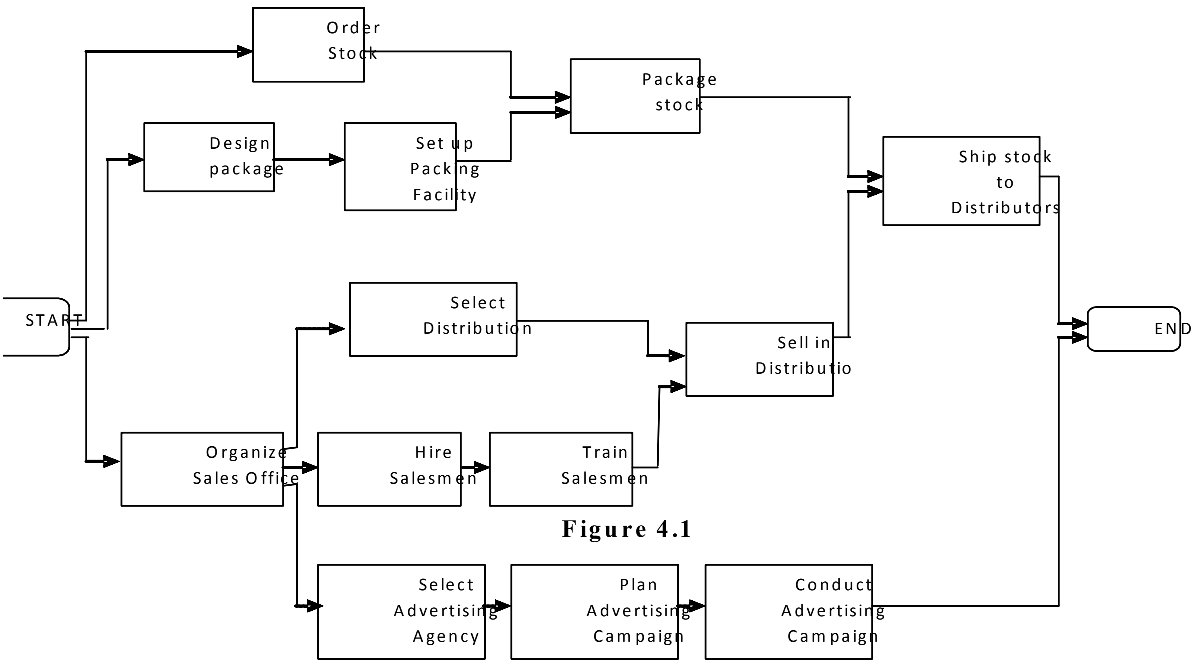


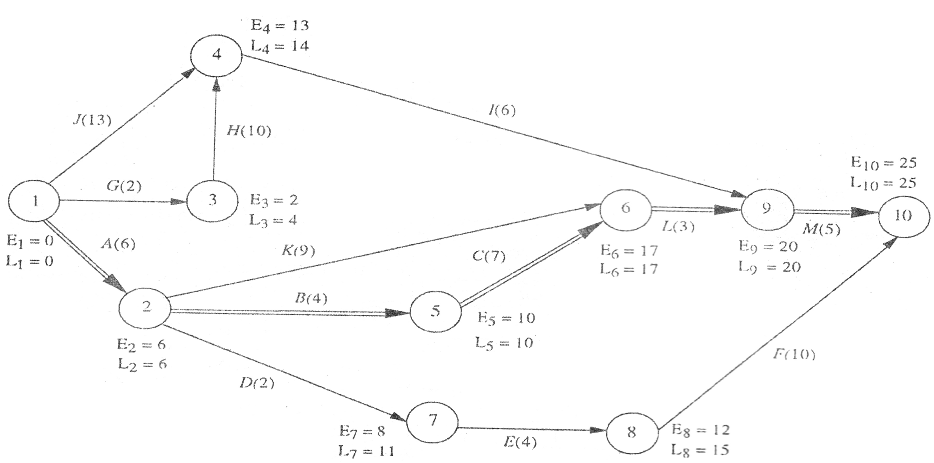
Figure 4.1

1. Draw the network diagram.

2. Indicate the critical path.

3. For each non critical activity, find the total and free float.

Solution:



Forward Pass Method:

$E_1 = 0$ $E_2 = E_1 + t_{1.1} = 0 + 6 = 6$

$E_3 = E_1 + t_{1.3} = 0 + 2 = 2$

$E_4 = \text{Max}\ (Ei + ti_{.4}) = \text{Max}\ (E_1 + t_{1.4};\ E_3 + t_{3.4})$

$= \text{Max}\ (0 + 13;\ 2 + 10) = 13$

$E_5 = E_2 + t_{2.5} = 6 + 4 = 10$

$E_6 = \text{Max}\ (Ei + ti_{.6}) = \text{Max}\ (E_2 + t_{2.6};\ E_5 + t_{5.6})$

$= \text{Max}\ (6 + 9;\ 10 + 7) = 17$

$E_7 = E_2 + t_{2.7} = 6 + 2 = 8$

$E_8 = E_7 + t_{7.8} = 8 + 4 = 12$

$E_9 = \text{Max}\ (E_i. + t_{i.9}) = \text{Max}\ (E_4 + t_{4.9;}\ E_6 + t_{6.9})$

$= \text{Max}\ (13 + 6;\ 17 + 3) = 20$

$E_{10} = \text{Max}\ (Ei + t_{i.\ 10}) = \text{Max}\ (E_8 + t\ 8._{10}.\ E_9 + t_{9.\ 10})$

$= \text{Max}\ (12 + 10;\ 20 + 5) = 25$

Backward Pass Method:

$L_{10} = E_{10} = 25$

$L_9 = L_{10} - t_{9.10} = 25 - 5 = 20$

$L_8 = L_{10} - t_{8.10} = 25 - 10 = 15$

$L_7 = L_8 - t_{7.9} = 15 - 4 = 11$

$L_6 = L_9 - T_{6.9} = 20 - 3 = 17$

$L_5 = L_6 - t_{5.6} = 17 - 7 = 10$

$L_4 = L_9 - t_{4.9} = 20 - 6 = 14$

$L_3 = l_4 - t_{3,4} = 14 - 10 = 4$

$L_2 = \text{Min}(L_j + t_{2.j}) = \text{Min}(L_5 - t_{2.5}\ L_6 - t_{2.6};\ L_7 - t_{2.7})$

$= \text{Min}(10-4;\ 17-9;\ 11-2) = 6$

$L_1 = \text{Min}(L_j + t_{1.j}) = \text{Min}(L_2 - t_{1,2};\ L_3 - t_{1.3};\ L_4 - t_{1,4})$

$= \text{Min}(6-6; 4-2; 14-13) = 0$

2. The critical path in the network has been shown by the double line by joining all those events where the two values E. and L. are equal. The critical path of the project is 1 -2-5-6-9-10 and the critical activities are A, B, C, L and M. The total project time is 25 weeks
3. For each non-critical activity, the total float and free float calculations are listed in the table below.

Activity	Duration (t I,j)	Earliest Time		Latest Time		Float		
		Start (Ei)	Finish (Ei + Uj)	Start (Lj-Uj)	Finish Lj	Total (Lj-Uj)-Ei	Free (Ej-Ei)-t i,j	
1-3	2	0	2	2	4	2	0	
1-4	13	0	13	1	14	1	0	
2-6	9	6	15	8	17	2	2	
2-7	2	6	8	9	11	3	0	
4-9	6	13	19	14	20	1	1	
7-8	4	8	12	11	15	3	0	
8-10	10	12	22	15	25	3	3	

Problem - 2: A banking company has decided to modernize one of its branch offices. The major activities of the project, along with the durations and preceding activities involved in the renovation process are listed in the table below:

Activity	A	B	C	D	E	F	G	H	I	J	K	L	M
Preceding activity	E	A	B	K	-	E	F	F	F	I,L	C,G,H	D	I,L
Duration (weeks)	4	2	1	12	14	2	3	2	4	3	4	2	2

1. Draw the network diagram
2. Find the minimum time in which the renovation is completed.

Solution:

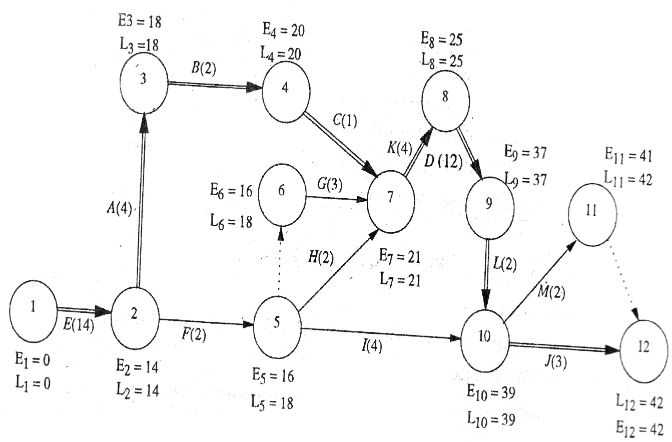


Figure 4.3

2. The critical path of the project is 1-2-3-4-7-8-9-10-12 and critical activities are E, A B, C, K, D, L and J. The total project time is 42 weeks.

Problem-3: Listed in the table are the activities and sequencing requirements necessary for the completion of a research work.

Activity	A	B	C	D	E	F	G	H	I	J	K	L	M
Preceding activity	-	-	B	C	A,D	D	A,D	E	G,H	I	G	J,K	L
Duration (weeks)	6	5	9	2	2	1	6	5	6	2	4	3	1

1. Draw the network diagram.
2. Find the critical path and duration of the project.
3. Find the total free and independent floats for various activities

Solution:

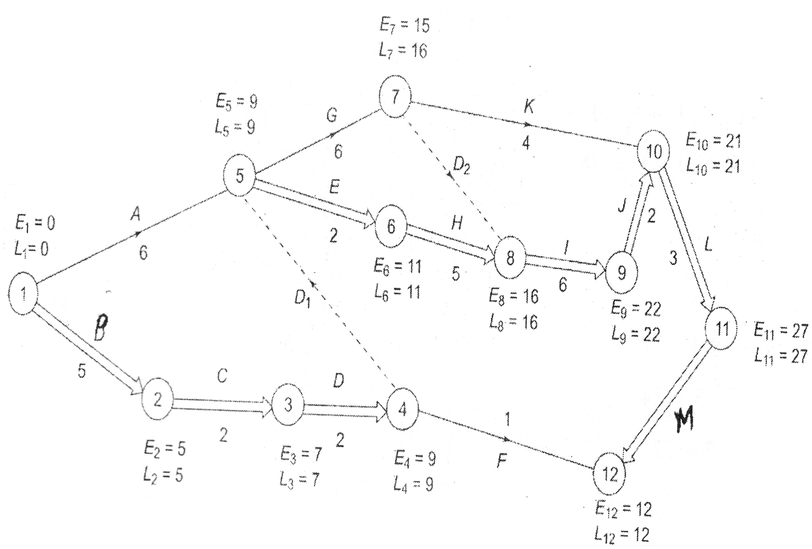


Figure 4.4

Forward Pass Method:

$E_1 = 0$

$E_2 = E_1 + t_{1,2} = 0+5=5$

$E_3 = E_2 + t_{2,3} = 5+2=7$

$E_4 = E_3 + t_{3,4} = 7+2=9$

$E_5 = \text{Max}\ (E_i + t_{i,4)}) = \text{Max}\ (E_1 + t_{1,5}\ ;\ E_4 + t_{4,5})$

$= \text{Max}\ (0+6;9+0)=9$

$E_6 = E_5 + t_{5,6} = 9+2=11$

$E_7 = E_5 + t_{5,7} = 9+6=15$

$E_8 = \text{Max}\ (E_7 + t_{7,8};\ E_6 + t_{6,8})$

$= \text{Max}\ (15+0;\ 11+5)=16$

$E_9 = E_8 + t_{8,9} = 16+6=22$

$E_{10} = \text{Max}\ (E_7 + t_{7,10};\ E_9 + t_{9,10})$

$= \text{Max}\ (15+4;22+2)=24$

$E_{11} = E_{10} + t_{10,11} = 24+3=27$

$E_{12} = \text{Max}\ (E_{11} + t_{11,12;}\ E_4 + t_{4,12})$

$= \text{Max}\ (27+1;9+1)=28$

Backward Pass

L12=28

L11=L12- 1=28-1= 27

L10=Lll-3 = 27-3 = 24

L9= L10 - 2 = 24-2 = 22

L8=L9-6 = 22-6=16

L7=Min(L10-4;L8-0)

Min (24-4; 16-0) = 16

L6 = L8-5 = 16-5 = 11

L5= Min (L5 - 6 ; L6 - 2)

Min (15 -6; 11 -2) = 9

L4 = Min(L12-l ;L5-0)

Min (28- 1 ; 9 - 0) = 9

L3= L4 - 2 = 9 - 2 = 7

L2 = L3 - 2 = 7- 2= 5

Ll= Min (L5 - 6 ; L2 - 5)

Min(9-5;5-5) = 0

The critical path is 1-2-3-4-5-6-8-9-10-11-12, the critical activities are B,C,D,E,H,I,J,L, and M.

The various float of each activity are calculated in table:

Activity	Duration	Earliest		Latest		Total Float	Free Float	Independent Float	
		Start	Finish	Start	Finish				
A	6	0	6	3	9	3	3	3	
B	5	0	5	0	5	0	0	0	
C	2	5	7	5	7	0	0	0	
D	2	7	9	7	9	0	0	0	
E	2	9	11	9	11	0	0	0	
G	6	9	15	10	16	1	0	0	
H	5	11	16	11	16	0	0	0	
I	6	16	22	16	22	0	0	0	
J	2	22	24	22	24	0	0	0	
K	4	15	19	20	24	5	5	4	
L	3	24	27	24	27	0	0	0	

Problem - 4:

A project schedule has the following features:

Activity	1-2	1-3	2-4	3-4	3-5	4-9	5-6	5-7	6-8	7-8	8-10	9-10
Duration (weeks)	4	1	1	1	6	5	4	8	1	2	5	7

1. Construct the network diagram
2. Compute the total float free float and independent float for each activity
3. Find the critical path and the total duration of the project.

Solution:

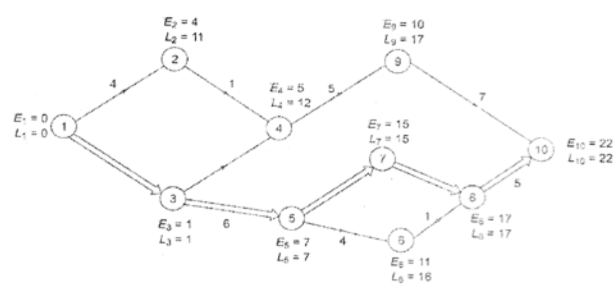


Figure 4.5

2. Critical Path is 1-3-5-7-8-10 and the project duration is 22 days.

3. The various float of each activity are calculated in table:

Activity	Duration	Earliest		Latest		Total Float	Free Float	Independent Float	–
		Start	Finish	Start	Finish				
1-2	4	0	4	7	11	7	0	0	
1-3	1	0	1	0	1	0	0	0	
2-4	1	4	5	11	12	7	2	0	
3-4	1	1	2	11	12	10	5	5	
3-5	6	1	7	I	7	0	0	0	
4-9	8	7	15	7	15	0	0	0	
5-6	4	7	11	12	16	5	0	0	
5-7	2	15	17	15	17	0	0	0	
6-8	1	11	72	16	17	5	5	0	
7-8	5	5-	10	12	17	7	0	0	
8-10	7	10	17	15	22	5	5	0	
9-10	5	17	22	17	22	0	0	0	

7.8 SUMMARY

The network diagram are constructed taking time required, proceeding activity and successive activity. The free time or slack time can be calcualated using the network. The managers can plan resources by checking the avaialable free time. For example activity 5-6 has 3 days free time and activity 5-7 is critical, then the labourers can be deputed to 5-7 instead of doing 5-6.

7.9 KEY WORDS

Free float

Independent float

Critical Path

Total Float

7.10 SELF ASSESSMENT QUESTIONS

Case Study-1: A project has the following schedule:

Activity	1-2	1-3	1-4	2-5	3-6	3-7	4-6	5-8	6-9	7-8	8-9
Duration (months)	2	2	1	4	8	5	3	1	5	4	3

1. Draw the network.
2. Calculate the total, free and independent float for each activity
3. Find the critical Path and the duration of the project.

Case Study-2: The R&D of SONY is developing a new power supply for a high definition television. The job is broken down into following form:

Job	Description	Predecessor job	Expected time (days)
A	Determine output voltage	-	5
B	Determine to use solid state rectifier	A	7
C	Choose Rectifier	B	2
D	Choose Filter	B	3
E	Choose Transformer	C	1
F	Choose Chassis	D	2
G	Choose Mounting	C	1
H	Layout Chassis	E,F	3
I	Build and Test	G,H	10

1. Draw a critical path scheduling arrow diagram, identify the critical path
2. What is the minimum time for completion of the job?

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UNIT 8 : DIFFERENT TIME ESTIMATES – P E R T

STRUCTURE

8.0 Objectives

8.1 Introduction

8.2 PERT with three time estimates

8.3 PERT Procedure

8.4 Illustrations

8.5 Summary

8.6 Key Words

8.7 Self Assessment Questions

8.8 References

8.0 OBJECTIVES

After studying this unit, you should be able to ;

* Discuss the three time estimates computation.
* Explain the procedure of computing PERT
* Examine the steps involved in the computing
* Determine the expected time of project completion.
* Solve the problems related to PERT and CPM

8.1 INTRODUCTION

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PERT (Programme Evaluation and Review Technique) is essentially a management technique and if tailored properly, can be used with advantage for responsibility accounting in addition to attaining other well defined objectives. Managers have found this technique for immense value where adopted judiciously and when configurations of events activities are correctly assessed and their times are realistically worked out.

PERT is designed for scheduling complex projects that involve many inter-related tasks. It improves the planning process because:

1. It helps the planner to define the project's various components activities and even logically.
2. It provides a basis for normal time estimates, and yet allows for some measure optimism or pessimism in estimating the completion dates.
3. It shows the effects of changes to the overall plan as they contemplated.
4. It provides a built-in means for on-going evaluation of the plan.
5. It facilitates the process of communication between planner's management by either; adhering to organizational lines or crossing over them. In essence, PERT makes the. \clear-cut assignment of responsibility possible.

8.2 PERT WITH THREE-TIME ESTIMATES

If the duration of activities in a project is uncertain, then activity scheduling calculations are done by using the expected value of durations However, such expected urallon estimations may not be given an accurate answer. Thus, rather than estimating directly the expected completion time of an activity, three values are considered. From these times a single value is estimated for future consideration. This is called three-time: estimates in PERT. The three times estimates are listed below:

Optimistic Time (t_o or a)

This is the shortest (minimum) possible time to perform an activity, assuming tha everything goes well.

Pessimistic Time (t_p or b)

This is the maximum time that is required to perform an activity, under extremely bad conditions.

Most Likely Time (t_m or m)

It refers to the estimate of the normal time the activity would take. This assumes normal delays. It is the mode of the probability distribution.

From these three time estimates, the expected time of an activity is calculated. It is given by the weighted average of the three time estimates.

$$t_e = \frac{t_o + 4t_m + t_p}{6}$$

Variance of the activity is given by $\sigma^2 = \left(\frac{tp - tp}{6}\right)^2$

8.3 PERT PROCEDURE

Step-1 : Draw the project network.

Step-2: Compute the expected time duration of each activity using $t_e = (t_o + 4\ t_m + t_p)/ 6$

Step-3: Compute the expected variance σ^2 of each activity.

Step-4: Compute the earliest start, earliest finish, latest start, latest finish and total float of each activity.

Step-5: Determine the critical path and identify the critical activities.

Step-6: Compute the expected variance of the project length σ^2 which is the sum of the variance of all the critical activities and hence find the standard deviation of the project length σ

Step-7: Compute the expected standard deviation of the project length Z= Ts - Tc /6

Where Ts = Specified or scheduled time to complete the project.

Tc = Normal expected duration of the project.

0 = Expected standard deviation of the project 'length'.

8.4 ILLUSTRATIONS

Problem-1:

A small project comprises of activities whose time estimates are given in table

Activity	Estimated Duration (Weeks)			
	Optimistic	Most Likely	Pessimistic	
1-2	1	1	7	
1-3	1	4	7	
1-4	2	2	8	
2-5	1	1	1	
3-5	2	5	14	
4-6	2	5	8	
5-6	3	6	15	

1. Draw the project network
2. Find the expected duration and variance of each activity. What is the expected project length?
3. Calculate the variance and standard deviation of the project length. What is the probability that the project will be completed

 1. At least 4 weeks earlier than expected?

 2. No more than 4 weeks later than expected time?

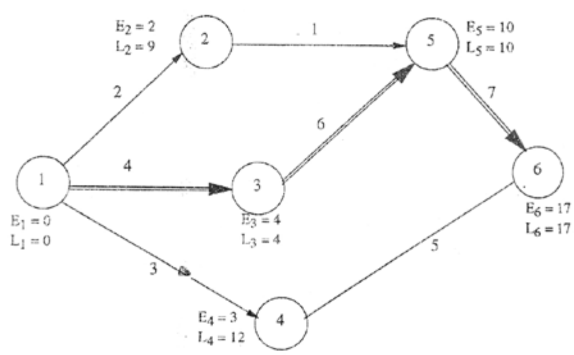
Solution:

The expected time and variance of each activity is calculated in the following table:

$$t_e = \frac{t_o + 4t_m + t_p}{6}$$

Variance of the activity is given by $\sigma^2 = \left(\frac{tp - tp}{6}\right)^2$

Activity	Optimistic Time (t_o)	Most Likely Time (t_m)	Pess Tim		
1-2	1	1	7	2	1
1-3	1	4	7	4	1
1-4	2	2	8	3	1
2-5	1	1	1	1	0
3-5	2	5	14	6	4
4-6	2	5	8	5	1
5-6	3	6	15	7	4



a) By examining all the paths we see that the critical path is 1-3-5-6.

b) The expected project length is the sum of the duration of each critical activity. That is duration of project = 4 + 6 + 7=17 weeks.

c) Variance of project length is the sum of the variance of the critical activities that is,

Variance of project length σ^2=l+4 + 4 = 9 weeks.

Standard Deviation, σ =? 9 = 3 weeks

1. Probability that the project will be completed at least 4 weeks earlier than expected time of 17 weeks is given by,

$$\text{Prob}\left[\frac{Z = T_s - T_e}{\sigma}\right] = \left[\frac{(17-4)-17}{3}\right] = \text{Prob.Z=-1.33}$$

But Z = - 1.33 from normal distribution table is 1- 0.9082 = 0.0918. Thus the probability of completing the project within 13 week (that is 4 week earlier) is 1-0.9082 = 0.0918= 9.18%. I

2. Probability that the project will be completed 4 weeks later than expected time of 17 weeks is given by,

$$\text{Prob.}\left[\frac{Z = T_s - T_e}{\sigma}\right] = \left[\frac{(17-4)-17}{3}\right] = \text{Prob.Z=-1.33}$$

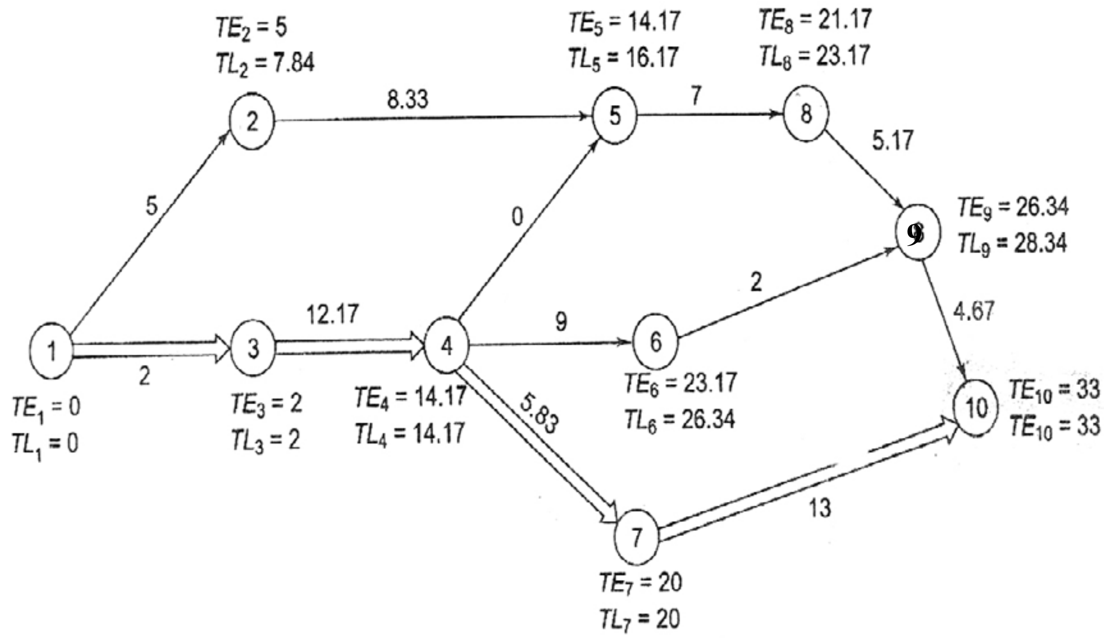
But Z =1.33 from normal distribution table is 0.9082. Thus the probability r completing the project within 21 week (that is 4 week later) is = 0.9082 = 90.82 %.

Problem-2: The following table gives the three estimates draw the network of the project and calculate the slack for each event. Find the critical path and the probability of completing the project in 35 days.

Activity	1-2	1-3	2-5	3-4	4-5	5-8	4-6	4-7	6-9	8-9	7-10	9-10
t_o	3	1	6	8	0	5	6	3	1	3	5	2
t_m	5	2	8	12	0	7	9	6	2	5	14	5
t_p	7	3	12	17	0	9	12	o	3	8	17	6

The expected time and variance of each activity is calculated in the following table:

Activity	Optimistic Time (t_0)	Most Likely Time (t_m)	Pessimistic Time (t_p)	te = ($to+4\ t_m+\ t_p$) /6	Variance σ^2
1-2	3	5	7	5	0.44
1-3	1	2	3	2	0.11
2-5	6	8	12	8.33	1
3-4	8	12	17	12.17	2.25
4-5	0	0	0	0	0
5-8	5	7	9	7	0.44
4-6	6	9	12	9	1
4-7	3	6	8	5.83	0.69
6-9	1	2	3	2	0.11
8-9	3	5	8	5.17	0.69
7-10	5	14	17	13	4
9-10	2	5	6	4.67	0.44



The critical path is 1-3-4-7-10.

The project duration is 33 days. That is Te = 33 days,

The expected variance of the critical path is = 0.11 + 2.25 + 0.69 + 4 - 7.05 days

Standard Deviation, $\sigma = \sqrt{7.05} = 2.65$ days

The probability of completing the project work in 35 days is,

Prob. $Z = \frac{Ts - Te}{\sigma} = \frac{35 - 33}{2.65} = 0.75$

For Z= 0.75 the normal distribution table gives a value of 0.7734. Thus the probability of completing the project in 35 days= 0.7734 = 77.34 %

The various floats are calculated as below :

Activity	Duration	Earliest		Latest		Total	Free	Independent
		Start	Finish	Start .	Finish	Float	Float	Float
1-2	5	0	5	2.84	7.84	2.84	0	0
1-3	2	0	2	0	2	0	0	0
2-5	8.33	5	13.33	7.84	16.17	2.84	0.840	0
3-4	12.17	2	14.17	2	14.17	0	0	0
4-5	0	14.17	14.17	16.17	16.17	2	0	0
5-8	7	14.17	21.17	16.17	23.17	2	0	0
4-6	9	14.17	23.17	17.34	26.34	3.17	0	0
4-7	5.83	14.17	20	14.17	20	0	0	0
6-9	2	23.17	25.17	26.34	28.34	3.17	1.17	0
8-9	5.17	21.17	26.34	23.17	28.34	2	0	0
7-10	13	20	33	20	33	0	0	0
9-10	4.67	26.34	31	28.33	33	2	2	0

Problem - 3: A project is represented by the following three time estimates:(in weeks)

Activity	A	B	C	D	E	F	G	H	I
t_o	5	18	26	16	15	6	7	7	3
t_p	10	22	40	20	25	12	12	9	5
t_m	8	20	33	18	20	9	10	8	4
Predecessor	-	-	-	A	A	B	C	D	EF

1) Draw the network and determine the critical path.
2) Expected task times and their variance.
3) The earliest and latest expected times to reach each event.
4) Probability of completing the project in 41.5 weeks

Solution:

Activity	Optimistic Time (t_0)	Pessimistic Time (t_p)	Most Likely Time (t_m)	t_e = (to+4 t_m+ t_p) /6	Variance σ^2
1-2	5	10	8	7.8	0.696
1-3	18	22	20	20	0.444
1-4	26	40	33	33	5.429
2-5	16	20	18	18	0.443
2-6	15	25	20	20	2.780
3-6	6	12	9	9	1.000
4-7	7	12	10	9.8	0.694
5-7	7	9	8	8	0.111
6-7	3	5	4	4	0.111

3. The earliest and latest expected time for each event will be calculated by considering the expected time of each activity.

Forward Pass:

$E_1=0$ $\quad$ $E2=E_1+t_{1,2}=0+7.8=7.8$

$E_3=E_1+t_{1,3}=0+20=20;$ $\quad$ $E_4=E_1+t_{1,4}=0+33=33$

$E_5=E_2+t_{2,5}=7.8+18=25.8;$ $\quad$ $E_6=\text{Max}\ (E_i+t_{i,6})$

$=\text{Max}\ (E_2+t_{2,6};\ E_3+t_{3,6})$

$=\text{Max}\ (7.8+20;20+9)=29$

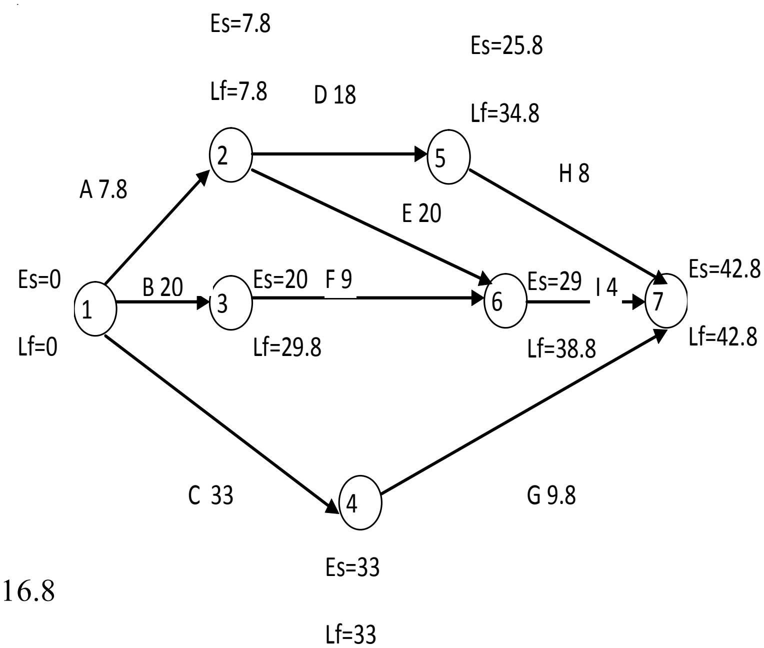
$E_7=\text{Max}\ (E_i+t_{i,7})$

$=\text{Max}\ (E_5+t_{5,7};\ E_6+t_{6,7};\ E_4+t_{4,7})$

$=\text{Max}\ (25.8+8;\ 29+4;33+9.8)$

$=42.8$

Backward Pass:



$L_7=E_7=42.8$

$L_6=L_7-t_{6,7}=42.8-4=38.8$

$L_5= L_7-t_{5,7}=42.8-8=34.8$

$L_4=L_7-t_{4,7}=42.8-9.8=33.0$

$L_3=L_6-t_{3,6}=38.8-9=29.8$

$L_2=\text{Min } (Lj-t_{2,j})$

$=\text{Min } (L_6-t_{2,6};\ L_5-t_{2,5})$

$=\text{Min } (38.8-20;\ 34.8-18) =16.8$

$L_1=\text{Min } (Lj-t_{1,j})$

$=\text{Min } (L_3-t_{1,3}; L_4-t_{1,4};\ L_2-t_{1,2})$

$=\text{Min } (29.8-20;\ 33-33;\ 16.8-7.8) =0$

4. The last event 7 will occur only after 42.8 weeks. For this we require only the duration of critical activities. This will help us in calculating the standard deviation of the duration of the last event.

Expected length of the critical path = 33 + 9.8 =42.8

Variance of the critical path - 5.429 + 0.694 = 6.123

Standard Deviation = σ = √ 6.123 = 2.474 weeks

The probability of finishing the project in 41.5 weeks is Z = Ts-Te = (41.5-42.8)/2.474

= -0.52

From the normal distribution table we have the value 1- 0.70 = 0.30 for Z= -0.52 that is the probability of completing the project in 41.5 weeks is 30%.

Problem -4: Consider the following project.

Activity	Three Time Estimates			Predecessor
	t_o	t_m	t_p	
A	3	6	9	-
B	2	5	8	-
C	2	4	6	A
D	2	3	10	B
E	1	3	11	B
F	4	6	8	C,D
G	1	5	15	E

Find the path and standard deviation. Also find the probability of completing the project by 18 weeks.

Solution: The expected time and the variance of each activity is calculated as shown in the table below:

Activity	Optimistic Time (t_0)	Most likely Time (t_p)	Pessimistic Time (t_m)	te (to+4 t_m+ t_p) /6	Variance σ^2
A	3	6	9	6	1
B	2	5	8	5	1
C	2	4	6	4	0.444
D	2	3	10	4	1.777
E	1	3	11	4	2.777
F	4	6	8	6	0.444
G	1	5	15	6	5.444

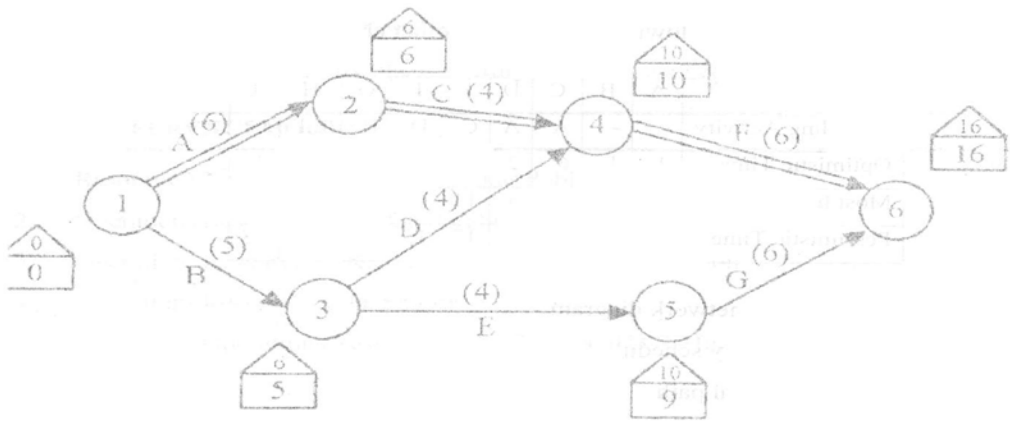


Figure 5.4

Critical Path is 1-2-4-6 or A-C- F

The project length = 6 + 4 + 6 = 16 weeks

Project length variance = $\sigma^2 = 1 + 0.444 + 0.444 = 1.888$

Standard Deviation $\sigma = \sqrt{1.888} = 1.374$ weeks

The probability of completing the project in 1 8 weeks

Prob. $Z = \frac{Ts - Te}{0} = \frac{18-16}{1.374} = 1.456$

From the normal distribution table we have the value 0.92647 for Z = 1.456 that is the probability of completing the project by 18 weeks is 92.65%.

8.5 SUMMARY

In this block you have gained a fair knowledge on network analysis. You have learnt to find out the total time required to complete project. You are also able to compute estimated time from different time estimates. Further you can learn about crashing of projects where in you can try to squeeze the project time by involving extra cost. The normal distribution tables you have learnt in block three is also used here.

8.6 KEY WORDS

PERT

Activity

Optimistic

Probability

Pessimistic

Projects

8.7 SELF ASSESSMENT QUESTIONS

Case Study-1:

A project has the following activities and other characteristics

Activity	A	B	C	D	E	F	G	H	I
Preceding Activity	-	-	A	A	C	D	B	E,F	G
Optimistic Time	4	1	6	2	5	3	3	1	4
Most likely Time	7	5	12	5	11	6	9	4	19
Pessimistic Time	16	15	30	8	17	15	27	7	28

1) Draw the PERT network diagram
2) Prepare the activity schedule for the project.
3) Identify the critical path
4) Determine the project completion time
5) Find the probability that the project is completed in 36 weeks.

Case Study-2: The owner of a retail outlet is considering a new computer system for transaction and inventory management. A computer company has sent the following instructions with regard to the installation of the system.

Activity	Activity	Immediate	Three time	estimates	
	Description	Predecessor	Optimistic	Most Likely	Pessimistic
A	Select the computer	-	4	6	8
B	Design Input/ Output system	A	5	7	15
C	Design monitoring system	A	4	8	12
D	Assemble computer hardware	B	15	20	25
E	Develop the main program	B	10	18	26
F	Develop the input/output routine	C	8	9	16
G	Create database	E	4	8	12
H	Install the system	D,F	1	2	3
I	Test and Implement	G,J	6	7	8

1. Construct the network diagram
2. Determine the critical path and expected completion time.
3. Determine the probability of completing the work in 5 5 days.

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UNIT 9 : INTRODUCTION TO PROBABILITY

STRUCTURE

9.0 Objectives

9.1 Introduction

9.2 Definitions

9.3 Three approaches to Probability

9.4 Set of Mutually Exclusive events

9.5 Probability Axioms

9.6 Theorems of Probability

9.7 Problems solved

9.8 Summary

9.9 Key Words

9.10 Self Assessment Questions

9.11 References

9.0 OBJECTIVES

After studying this unit you should able to :

* Explain the basic concepts of probability;
* Describe probability axioms and
* Discuss Theorems of probability.

9.1 INTRODUCTION

The concept of probability is the chance that something happens or will not happen. In statistics it is denoted by the capital letter **P** and is measured on an inclusive numerical scale of 0 to 1. If we are using percentages, then the scale is from 0% to 100%. If the probability is 0% then there is absolutely no chance that an out come will occur.

The opposite of probability is **deterministic** where the outcome is certain on the assumption that the input data is reliable. With probability something happens or it does not happen, that is the situation is **binomial**, or there are only two possible outcomes. However that does not mean that there is a 50/50 chance of being right or wrong or a 50/50 chance of winning. If you toss a fair-sided coin, one that has not been "fixed", you have a 50% chance of obtaining heads or 50% chance of throwing tails. If you buy one ticket in a fund raising raffle then you will either win or lose.

9.2 BASIC DEFINITIONS

Before we give definitions of the word probability as looked upon by various schools of thought it is necessary that we familiarise ourselves with certain terms that are used in this context.

Random Experiment : It is an experiment which if conducted repeatedly under homogeneous conditions does not give the same result**.** The result may be anyone of the various possible 'outcomes'. Here the result is not unique (or the same every time). For example if an unbiase**d** dice is thrown it will not always fall with any particular number up. Any of the six numbers on the dice can come up.

Trial and Event : The performance of a random experiment is called a *trial* and the outcome an *event.* Thus throwing of a dice would be called a trial and the result (falling of anyone of the six numbers I, 2, 3, 4, 5, 6) **an event.**

Events could be either simple or compound (also called composite). An event is called simple if it corresponds to a single possible outcome. Thus in tossing a dice, the chance of getting 3 is a simple event (because 3 occur in the dice only once). However the chance of getting an odd number is a **compound event** (because odd numbers are more than one i.e. 1, 3 and 5).

A compound, event can further be decomposed into simple events. e.g. of a die 5 thrown, getting an odd number 5 an compound event but it can be decomposed into simple events as getting (1, 3 or 5) i.e. there are three simple events for the above stated compound event.

Exhaustive Cases: All possible outcomes of an event are known as exhaustive cases. In. the throw of a single dice the exhaustive cases are 6 as the dice has only six faces each marked with a different number. However if 2 dice are thrown the exhaustive cases would be 36 (6 x 6) as there are 36 ways in which two dice can fall. Similarly the number of exhaustive cases in the throw of 2 coins would be four (2 x 2) i.e. *HH, TT*, *HT* and *TH,* (where *H* stands for head and *T* for tail).

Favourable Cases : The number of outcomes which result in the happening of a desired event are called favourable cases. Thus in a single throw of a dice, the number of favourable cases of getting an odd number are three i.e. 1, 3 and 5. Similarly in drawing a card from a pack, the cases favourable to getting a spade are 13 (as there are 13 spade cards in the pack).

Mutually Exclusive Events: Two or more events are said to be mutually exclusive if the happening of anyone of them excludes the happening of all others in a single (i.e. same) experiment. Thus in the throw of a single dice the event 5 and 6 are mutually exclusive because if the event 5 happens no other event is possible in the same experiment. Here *one and only one* of the events can take place at a time excluding all others.

Equally Likely Events: Two or more events are said to be equally likely if the chance of their happening is equal i.e., there is no preference of anyone event over the other. Thus in a throw of an unbiased die, the coming up of 1, 2, 3, 4, 5 or 6 is equally likely. In the throw of an unbiased coin the coming up of head or tail is equally likely.

Independent and Dependent Events: An event is said to be independent if its happening is not affected by the happening of other events and if it does not affect the happening of other events. Thus in the throw of a dice repeatedly, coming up of 5 on the first throw is independent of coming up of 5 again in the second throw.

However if we are successively drawing cards from a pack (without replacement) the events would be dependent. The chance of getting a King on the first draw is 4/52 (as there are 4 Kings in a pack). If this card is not replaced before the second draw, the chance of getting a King again is 3/51 as there are now only 51 cards left and they contain only 3 Kings.

If however the card is replaced after the first draw i.e. before the second draw the events would remain independent. In each of the two successive draws the chance of getting a King would be 4/52.

(i) The number of permutations of n dissimilar things taken all at a time is n!. Thus if there are 3 letters A, *B and* C, the total number of ways in which they can be arranged is *ABC, ACB, BAC, BCA, CAB* and *CBA* i.e. 3! = 3x 2x 1 = 6.

Factorial *n* (written as *n!)* is equal to the continued product of *n* natural numbers starting from 1 i.e.

$$n! = 1 \times 2 \times 3 \ldots\ldots\ldots (n-1)\, n$$

$$= n\,(n-1)\,(n-2) \ldots\ldots\ldots 3.\,2\,.\,1$$

$$= n\,(n-1)\,! = n\,(n-1)\,(n-2)\,!$$

(ii) The number of permutations of *n* dissimilar things taken *r* at a time is $^nP_r = \frac{n!}{(n-r)!}$.

Thus if we are to make arrangements of any two letters out of three letters A, *B,* C, then the different arrangement will be *AB, BA,* A C, *CA, BC, CB* i.e. 6 arrangements which in factorial notation can be represented as

$$^3P_2 = \frac{3!}{(3-2)!} = 3! = 3 \times 2 \times 1 = 6$$

$\Rightarrow$ Three letters taken 2 at a time can be arranged in 6 ways. The number of arrangement of any two letters out of 4 = $^4c_2 = \frac{4!}{2!} = 4 \times 3 = 12$

(iii) The number of permutations of *n* things when n_1 of them are of one kind and n_2 of another kind is $\frac{n!}{n_1!n_2!}$ Thus if we have to find out the permutations of the letters of the word FARIDABAD (where A occurs 3) times and D occurs 2 times) the answer would be $\frac{9!}{3!2!}$ or $\frac{9\times8\times7\times6\times5\times4\times3\times2\times1}{3\times2\times1\times2\times1} = 30{,}240$

(iv) The fundamental rule of counting is that if an operation can be performed in '*m*' ways and having been performed in any one of these ways a second operation can be performed in '*n*' ways, the total number of ways of performing the two operations together is *m* x *n*.

v) The number of combination of *n* different things taken *r* at the time is $^nC_r \frac{n!}{r!(n-r)!}$. Thus if we have to pick up two alphabets out of three, A, three A, B and C, we can pick up AB, or AC or BC i.e 3 ways or $^3C_2 = \frac{3!}{2!(3-2)!} = \frac{3!}{2!} = \frac{3\times2\times1}{2} = 3$.

9.3 THREE APPROACHES TO PROBABILITY

Subjective probability

One type of probability is **subjective probability**, which is qualitative, sometimes emotional, and simply based on the belief or the "gut" feeling of the person making the judgment.

Subjective probability may be a function of a person's experience with a situation. For exam- ple, Salesperson A says that he is 80% certain of making a sale with a certain client, as he knows the client well. However, Salesperson B may give only a 50% probability level of making that sale. Both are basing their arguments on subjective probability.

Relative frequency probability

A probability based on information or data collected from situations that have occurred previously is **relative frequency probability**

Relative frequency probabilities have use in many business situations. For example, data taken from a certain country indicate that in a sample of 3,000 married couples under study, one-third were divorced within 10 years of marriage. Again, on the assumption that future conditions will be similar to past conditions, we can say that in this country, the probability of being divorced before 10 years of marriage is 1/3 or 33.33%. This demographic information can then be extended to estimate needs of such things as legal services, new homes, and child- care.

Classical probability

A probability measure that is also the basis for gambling or betting, and thus useful if you frequent casinos, is **classical probability**. Classical probability is also known as **simple probability** or **marginal probability** and is defined by the following ratio:

In order for this expression to be valid, the probability of the outcomes, as defined by the numerator (upper part of the ratio) must be equally likely.

9.4 SET OF MUTUALLY EXCLUSIVE EVENTS

To cover all possibilities between mutually exclusive events add up all the probabilities. Probabilities of all these events together add up to 1.

$$P(A) + p(B) + p(C) + \ldots\ldots p(N) = 1$$

Exhaustive Events

A happens or A does not happen then A and B are Exhaustive Events.

$$P(A \text{ happens}) + A (\text{does not happen}) = 1$$

The Sum of the probabilities of all mutually exclusive and collective exhaustive events is always equal to 1. That is,

$$P(A) + p(B) + p(C) = 1$$

If A,B,Care mutually exclusive and collective event.

Example 1 : -

$$P(\text{You Pass}) = 0.9$$

$$P(\text{you fail}) = 1\ 0.9 = 0.1$$

EXAMPLE 1 – EXHAUSTIVE EVENTS

A production line uses 3 machines. The Chance that 1^{st} machine breaks down in any week is 1/10. The chance for 2^{nd} machine is 1/20. Chance of 3^{rd} machine is 1/40. What is the chance that least one machine working in any week?

Solution

P(at least one not working) + p(all the three working) =1

P(at least one not working) + 1-p(all the three working)

P(all three working) = p(1^{st} working) * p(2^{nd} working) * p(3^{rd} working)

p(1^{st} working)=1- p(1^{st} not working) = 1-1/10=9/10

p(2^{nd} working) = 19/20

p(3^{rd} working) = 39/40

p(working)9/10 *19/20*39*40= 6669/8000

p(at least 1working) =1-6669/8000= 1331/8000

9.5 PROBABILITY AXIOMS

We now look at probability axioms. These are general probability rules that hold regardless of the particular situation or kind of probability (objective or subjective). Given an experiment.

1. Each elementary event or a combination of elementary events, must have associated with it a probability greater than or equal to zero but less than or equal to 1. Thus, if A is an event within a sample space, then

$$P(A) = 1$$

2. The probability of an entire sample space is 1. Thus, if S represents a entire sample space, then

$$P(S) = 1$$

3. probability that one or the other or both of two mutually exclusive events will occur is equal to the sum of the individual probabiliteis of these events. Thus.

$$P(A \text{ or } B) = P(A) + P(B)$$

when A and B are mutually exclusive events.

4. The probability of an event that does not occur is equal to 1 minus the probability of the event that occurs. Thus

$$P(\overline{A}) = 1 - P(A)$$

where $\overline{A}$ is the non-occurrence of event A.

Example 1: Suppose we have a box with 3 red, 2 black and 5 white balls. Each time a ball is drawn, it is returned to the box. What is the probability of drawing:

a) Either a red or a black ball?

b) Either a white or a black ball?

Solution: The probabilities of drawing the specific colour ball are

P (red) = 0.3 P (black) = 0.2 P (white) = 05

Applying the rule 2, we find

P (red) + P (black) + P (white) = 0.3 + 0.2 + 0.5 = 1

As we want to know the probability of drawing either a red or a black ball, then the answer will be probability P (red) + P (black) = 0.3 + 0.2 = 0.5. Likewise, the probability of getting either a white ball or a black ball will be

$$P\ (\text{white}) + P\ (\text{black}) = 0.5 + 0.2 = 0.7$$

9.6 THEOREMS OF PROBABILITY

There are two important theorems of probability viz.,

(i) The addition theorem and

(ii) The multiplication theorem

Addition theorem

If A and B an any two events then the probability that at least one of them occurs in denoted by P ($A \cup B$) and is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Where P(A) = Probability of the occurrence of event A

P(B) = Probability of the occurrence of event B

P ($A \cap B$) = Probability of simultaneous occurrence of events A and B

Mutually exclusive events have no sample point common to them, therefore if *A* and *B* are two mutually exclusive events then $A \cap B = \phi$ i.e. the intersection of two mutually exclusive events is a null set and in this case $P(A \cap B) = 0$

$\therefore$ In case of mutually exclusive events

$$P(A \cup B) = P(A) + P(B)$$

If there are three events A, B and C. The probability of the occurrence of at least one of them 'is given by

$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$
$$- P(A \cap B) - P(B \cap C) - P(A \cap C)$$
$$+ P(A \cap B \cap C)$$

If the events are mutually exclusive then

$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$

In case of finite number say *n* of mutually exclusive events

$$P(A_1 \cup A_2 \cup A_3 \cdots \cdots \cup A_n) = P(A_1) + P(A_2) + \ldots\ldots + P(A_n)$$

Note : (i) If a number of events A_1, A_2 ...... A_n are mutually exclusive and Exhaustive then the sum of the individual probabilities of their happenings is equal to 1 i.e.

$$P(A_1) + P(A_2) + \ldots\ldots + P(A_n) = 1$$

(ii) If the events are finite and mutually exclusive then the probability of the occurrence of at least one of them is equal to the sum of their individual probabilities.

(iii) The event A and its compliment $\overline{A}$ can be considered as mutually exclusive and exhaustive.

$$\therefore P(A) + P(\overline{A}) = 1 \Rightarrow P(\overline{A}) = 1 = P(A)$$

9.7 PROBLEM SOLVED

Example 1

If A, B, C are mutually exclusive and exhaustive events, find P(B) if $\frac{1}{3}P(C) = \frac{1}{2}P(A) = P(B)$

Solution

Since the events are mutually exclusive and exhaustive,

P(A) + P(B) + P(C) = 1

Let $\frac{1}{3}P(C) = \frac{1}{2}P(A) = P(B) = k$

$$\Rightarrow P(C) = 3k, P(A) = 2k, P(B) = k$$

$$\therefore k + 2k + 3k = 1 \Rightarrow k = \frac{1}{6}$$

$$\therefore P(B) = \frac{1}{6}, P(A) = \frac{1}{3,} P(C) = \frac{1}{2} \text{ Ans.}$$

Example 2

One tickets is drawn at random from a bag containing 30 tickets numbered from 1 to 30. Find the probability that,

(a) It is a multiple of 5 or 7
(b) It is a multiple of 3 or 5

Solution

One ticket can be drawn out of 30 in ${}^{30}C_1$ = 30 ways. This is the total number of ways in which the event can take place, or it is the Exhaustive number of cases.

(a) Multiples of 5 are 5, 10, 15, 20, 25, 30.

Multiples of 7 are 7, 14, 21, 28.

Thus there 'are 6 multiples of 5 and 4 multiples of 7. None of these are common. So the events are mutually exclusive. The probability of having a multiple of 5 or 7 would be,

$$\frac{6}{30}+\frac{4}{30}=\frac{10}{30}=\frac{1}{3}$$

Multiples of 3 are 3, 6, 9, 12, 15, 18,21, 24, 27, 30

Multiples of 5 are 5, 10, 15, 20, 25, 30.

Thus there are 10 multiples of 3 and 6 multiples of 5. However two digits 15 and 30 are common to both sets. Therefore the events are not mutually exclusive. The required probability

$$=\frac{10}{30}+\frac{6}{30}-\left(\frac{10}{30}\times\frac{6}{30}\right)$$

$$=\frac{16}{30}-\frac{2}{30}=\frac{14}{30}=\frac{7}{15} \text{ Ans.}$$

Example 3

The probability that A will live upto 60 years is $\frac{3}{4}$ and probability that *B* will live up to 60 years is $\frac{2}{3}$. What is the probability (i) that both A and *B* will live upto sixty years (ii) that both die before reaching 60 years.

Solution

The events in equation are independent of each other and the rule of multiplication would be applied.

(i) The probability that both A and B live up to 60 years or *P(A* and *B)* = *P(A)* x $P(B) = \frac{3}{4}\times\frac{2}{3}=\frac{6}{12}=\frac{1}{2}$

(ii) Since the probabilities of the survival of A and B up to 60 years are $\frac{3}{4}$ and $\frac{2}{3}$ respectively, therefore the probabilities of their death would be $\left(1-\frac{3}{4}\right)=\frac{1}{4}$ and $\left(1-\frac{2}{3}\right)=\frac{1}{3}$ respectively.

(iii) Now the probability that both A and B would die before reaching 60 years, would be $\frac{1}{4}\times\frac{1}{3}=\frac{1}{12}$ Ans.

Example 4

A bag contains 4 white and 6 red balls. Two draws of one balls each are made without replacement. What is the probability that (i) one is red and the other white (ii) Both the balls are red.

Solution

Probability of drawing a red ball in the first draw or $P\ (A) = \frac{6}{10}$

Probability of drawing a white ball in the second draw given that the first draw has given a red ball or $P\ (B/\mathrm{A}) = \frac{4}{9}$ (since only 9 balls are left in the bag and four white balls are still there)

Probability of the combined event

$$P(AB)\ P(A \cap B) = P(A) \text{ x } P\ (B/A) = \frac{6}{10} \times \frac{4}{9} = \frac{24}{90}$$

But it could also happen that in the first draw a white ball was drawn then,

Probability of drawing a white ball in the first draw or $P\ (A) = \frac{4}{10}$ and

Probability of drawing a red ball in the second draw given that the first draw gave a white ball or $P(B/A) = \frac{6}{9}$

The combined probability of the two events is

$$P(AB) \text{ or } P(A \cap B) = P(A) \text{ x } P\ (B/A) = \frac{4}{10} \times \frac{6}{9} = \frac{24}{90}$$

Now anyone of the two situations (when we draw a red ball first or we draw a white .ball first), would satisfy the conditions of the problem. These two events are mutually exclusive. So the probability that anyone of the two happens is the sum of the two probabilities.

$$= \frac{24}{90} + \frac{24}{90} = \frac{48}{90} = \frac{8}{15} \text{Ans.}$$

Here we have applied both, the rule of multiplication and the rule of addition of probability.

However such problems can be very easily solved with rules of permutations and combinations.

Thus:

(i) Two balls can be drawn one of 10 balls in ${}^{10}C_2$ or $\frac{10!}{2!8!}$ or $\frac{10\times 9}{2}$ or 45 ways.

(ii) One white ball can be drawn out of 4 white balls in ${}^{4}C_1$ or $\frac{4!}{1!3!}$ or 4 ways.

(iii) One red ball can be drawn out of 6 red balls in ${}^{6}C_1$ or 6 ways.

(iv) The total number of ways of drawing a white and a red ball are ${}^{4}C_1 \times {}^{6}C_1$ or 4 x 6 = 24

(v) The required probability would be

$$= \frac{\text{No. of cases favourable the event}}{\text{Total No. of ways in which the event can happen}} = \frac{24}{45} = \frac{8}{15}$$

(ii) Required Probability = $\frac{{}^{6}C_2}{{}^{10}C_2}$

Example 5

(a) A committee of 4 persons is to be appointed from 3 officers of the production department, 3 officers of the sales department and 2 officers of the purchase department and 1 cost accountant. Find the probability of forming a committee in the following manner.

(i) There must be on from each category

(ii) It should have at least one from the purchase department

(b) If P(A) = 0.4, P(B) = 0.7 and P (at least one of A and B) = 0.8, find P (only one of A and B).

Solution:

Total number of ways of forming a committee of 4 persons = ${}^{9}C_4$

(i) Required Probability = $\frac{{}^{3}C_1 \times {}^{3}C_1 \times {}^{2}C_1 \times 1}{{}^{9}C_4}$

$$= \frac{18}{\frac{9\times 8\times 7\times 6}{4\times 3\times 2\times 1}} = \frac{1}{7}$$

(i) Required Probability

= 1 – Probability that nobody is taken from the purchase department

$$=1-\frac{{}^7C_4}{{}^9C_4}=1-\frac{\frac{7!}{4!3!}}{\frac{9!}{4!5!}}$$

$$= 1-\frac{7!}{3!}\times\frac{5!}{9!}=1-\frac{20}{72}=\frac{52}{72}=\frac{13}{18}$$

Second method

Required Probability

= Prob. of taking one from Purchase deptt. And three others + Prob. of taking 2 from Purchase deptt. and two others

$$=\frac{{}^2C_1\times{}^7C_3}{{}^9C_4}+\frac{{}^2C_2\times{}^7C_2}{{}^9C_4}$$

Required Probability = P(A-B) + P(B-A)

= P(A) + P(B) -2P $(A\cap B)$

P(A) + P(B) – 2 [P (A) + P(B) – $(A\cup B)$]

= P(A) – P(B) + 2 $(A\cup B)$

= 0.4 +0.7 – 2 x 0.8 = 0.5 Ans.

$P(A\cup B) = P(A) + P(B) - P(A\cap B)$

$\Rightarrow 0.8 = 0.4 + 0.7 - P(A\cap B)$

$P(A\cap B) = 0.3$? P(A) P(B)

Hence the events are not independent.

Examples 6

A husband and a wife appear in an interview for two vacancies in the same post. The probability of husband's selection is 1/7 and that of wife's selection is 1/5. What is the probability that:

(a) Both of them will be selected ;
(b) Only one of them will be selected;
(c) None of them will be selected

Solution:

Let A and B denote the events of husband's and wife's selection respectively.

Let $P(A) = p_1$; $P(B) = p_2$

$$\therefore p_1 = \frac{1}{7}, p_2 = \frac{1}{5}$$

P($\overline{A}$) = Probability of husband's rejection

$$= 1 - p_1 = 1 - \frac{1}{7} = \frac{6}{7} = q_1$$

P($\overline{B}$) = Probability of husband's rejection

$$= 1 - p_2 = 1 - \frac{1}{5} = \frac{4}{5} = q_2$$

Probability that husband and wife both are selected = P(A) P (B)

$$= \frac{1}{7} \times \frac{1}{5} = \frac{1}{35} \text{ Ans.} \qquad [\because \text{ the events are independent}]$$

(ii) Probability that only one of them will be selected

$$= P(A)\, P(\overline{B}) + P(\overline{A})\, P(B)$$

$$= \frac{1}{7} \times \frac{4}{5} \times \frac{6}{7} \times \frac{1}{5} = \frac{10}{35} = \frac{2}{7} \text{ Ans.}$$

(iii) Probability that both are rejected

$$= P(\overline{A})\, P(\overline{B}) = \frac{6}{7} \times \frac{4}{5} = \frac{24}{35} \text{ Ans.}$$

9.8 SUMMARY

Probability measures the likliness of occurance of an event. The out comes may have equal chances in some cases where as in some other cases, it may not be so.

9.9 KEY WORDS

Random Experiments

Trail Event

Exhaistive Cases

Mutually Exclusive Events

9.10 SELF ASSESSMENT QUESTIONS

1. A sub – committee of 6 members is to be formed out of group consisting of 7 men and 4 ladies. Calculate the probability that the sub-committee will consist of (1) exactly 2 ladies, and (ii) at least 2 ladies.
2. There are 3 economists, 4 engineers, 2 statisticians and I doctor. A committee of 4 from among them is to be formed. Find the probability that the committee.
   (i) Consist one of each kind
   (ii) Has at least one economist
   (iii) Has the doctor as a member and three others.
3. (a) a bag contains 6 white. 4 red and 10 black balls. Two balls are drawn at random. Find the probability that they will both be black.

   (b) A bag contains 8 white and 4 red balls. Five balls are drawn at random. What is the probability that 2 of them are red and 3white?
4. From a pack of 52 cards are drawn at random. Find the probability that one is king and other a queen?
5. One bag contains 4 white and 2 black balls. Another contains 3 white and 5 black balls. If one ball is drawn from each bag. Find the probability that

   (a) both are white, (b) both are black, and (c) one is white and one is black.

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UNIT -10 :PROBABILITY - TYPES

STRUCTURE

10.0 Objectives

10.1 Introduction

10.2 Marginal Probability

10.3 Joint Probability

10.4 Conditional Probabilities

10.5 Bayes' theorem

10.6 Problems solved

10.7 Summary

10. 8 Key Words

10.9 Self Assessment Questions

10.10 Reference

10.0 OBJECTIVES

After studying this unit you should be able to :

* Define marginal probability;
* Analyze joint Probability;
* Describe conditional probability and
* Explain Bayes' Theorem.

10.1 INTRODUCTION

Conditional probabilities are contingent on a previous result. For example, suppose you are drawing three marbles - red, blue and green - from a bag. Each marble has an equal chance of being drawn. What is the conditional probability of drawing the red marble after already drawing the blue one? First, the probability of drawing a blue marble is about 33% because it is one possible outcome out of three. Assuming this first event occurs, there will be two marbles remaining, with each having a 50% of being drawn. So, the chance of drawing a blue marble after already drawing a red marble would be about 16.5% (33% x 50%).

Definition of 'Conditional Probability'

Probability of an event or outcome based on the occurrence of a previous event or outcome. Conditional probability is calculated by multiplying the probability of the preceding event by the updated probability of the succeeding event.

Probability under Conditions of Statistical Independence:

When a statistically independent event occurs, it does not have any effect on the happening of another event. There are three types of probabilities under statistical independence: 1. Marginal, 2. Joint and 3. Conditional.

10.2 MARGINAL PROBABILITY

Marginal probability is the simple probability of the occurrence of an event. Right in the beginning, we have given such examples pertaining to the tossing of a coin. When a coin is tossed, the probability of getting a head is 0.5, so also in the case of getting a tail. These are known as marginal probabilities as a toss of a fair coin is a statistically independent event.

10.3 JOINT PROBABILITIES

The probabilities of two or more independent events occurring together is the product of their marginal probabilities.

Symbolically, P (AB) = P(A) x P(B)

Where P (AB) = probability of events A and B occurring together or in succession

P (A) = marginal probability of event A

P (B) = marginal probability of event B

It will be seen that this is the multiplication rule for joint, independent events.

Example: Suppose we toss a fair coin twice. What is the probability of getting two successive heads?

Solution: $P(H_1H_2) = P(H_1) \times P(H_2) = 0.5 \times 0.5 = 0.25$

Obviously, the probability of getting two successive tails is also the same. Since P(T) = P(H) = 0.5. If there are three tosses of a fair coin, then the joint probability of getting three successive heads will be

$P(H_1H_2H_3) = P(H_1) \times P(H_2) \times P(H_3) = 0.5 \times 0.5 \times 0.5 = 0.125$

Example : Suppose we have an unfair coin that has P(H) = 0.7 and P(T) = 0.3. What is the probability of getting three successive heads on tossing the coin three times?

Solution : $P(H_1H_2H_3) = P(H_1) \times P(H_2) \times P(H_3) = 0.7 \times 0.7 \times 0.7 = 0.343$

Again, we ask : What is the probability of getting three successive tails on tossing the coin three times?

$P(T_1 T_2 T_3) = P(T_1) \times P(T_2) \times P(T_3) = 0.3 \times 0.3 \times 0.3 = 0.027$

It may be noted that the two joint probabilities add up to 0.343 + 0.027 = 0.37 only and not to 1. This is because the events H_1 H_2 H_3 and T_1 T_2 T_3 do not form a collectively exhaustive list, though they are mutually exclusive in the sense that if one event occurs then the other event cannot occur. Let us take a few more examples.

Example : What is the probability of getting tail, head, and tail on three successive tosses of a fair coin?

Solution : $P(T_1H_2T_3) = P(T_1) \times P(H_2) \times P(T_3) = 0.5 \times 0.5 \times 0.5 = 0.125$

We will get the same answer in the case of joint probability of $P(H_1T_2H_3)$

Example : What is the probability of getting at least two tails on three successive tosses of a fair coin?

Solution : Now the sequence of three tosses could be one of the following :

$H_1H_2H_3$ $H_1T_2H_3$
$H_1\ T_2\ T_3$ (two tails) $T_1\ H_2\ H_3$
$T_1\ T_2\ H_3$ (two tails) $T_1\ T_2\ T_3$ (three tails)
$H_1\ H_2\ T_3$ $T_1\ H_2\ T_3$ (two tails)

Out of the total eight outcomes, we find that at least two tails occur four times. As the probability of any of the three successive tosses is 0.5 probability of getting at least two tails is

$P(H_1\ T_2\ T_3) + P(T_1\ T_2\ H_3) + P(T_1\ T_2\ T_3) = 0.125 + 0.125 + 0.125 + 0.125 = 0.5$

We will get the same answer in the case of the joint probability of at least two heads in three successive tosses.

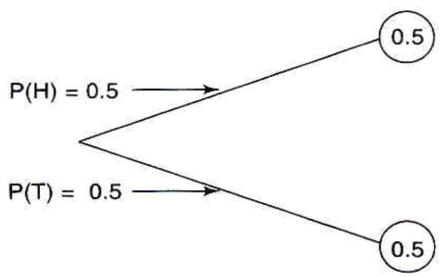
Example: What is the probability of getting three heads or three tails on three successive tosses?

Solution: $P(H^1\ H^2\ H^3 \text{ or } T^1\ T^2\ T^3) = P(H^1\ H^2\ H^3) + P(T^1\ T^2\ T^3)$

$= 0.125 + 0.125 = 0.25$

Since there can be only eight outcomes of which only one can be three successive heads and one can be three successive tails, each outcome has a joint probability of 0.125 as the total eight outcomes must equal to 1.

Probability Tree Diagrams: In order to have a better understanding of these examples, we may construct a probability tree. Figure 1 shows the outcome of tossing a fair coin once. We can have only two possible outcomes – a head or a tail.



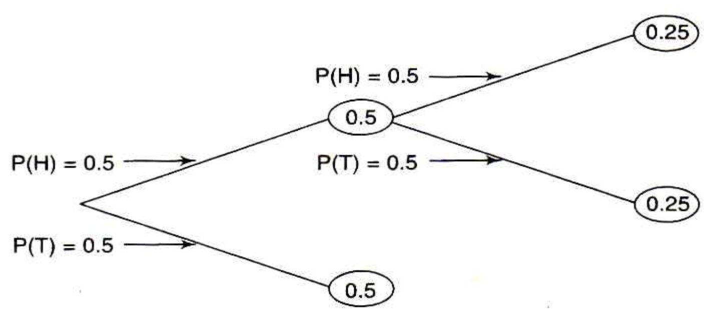


Fig 1 Example of Probability Fig 2 Probability tree of a Partial Second Toss Tree

Assuming that the first toss is a head, then in the second toss the outcome could be either a head or a tail. This is shown in Fig 2.

Now, we assume that the outcome of first toss is tail. In this situation, the second toss must originate from tail. This provides two more branches to the three as shown in Fig 3.

We may further extend the tree to depict the outcomes of the third toss. We repeat the same process, as a result we get what is depicted in Fig 4

It may be noted that when we toss once, we have two possible outcomes, when we toss a coin twice, we have four possible outcomes and when we toss it thrice, then we have eight possible outcomes.

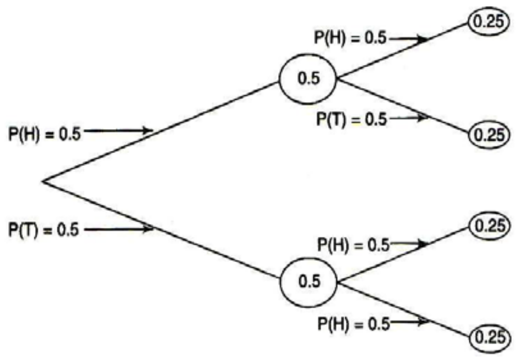


Fig 3 : Probability Tree of Two Tosses

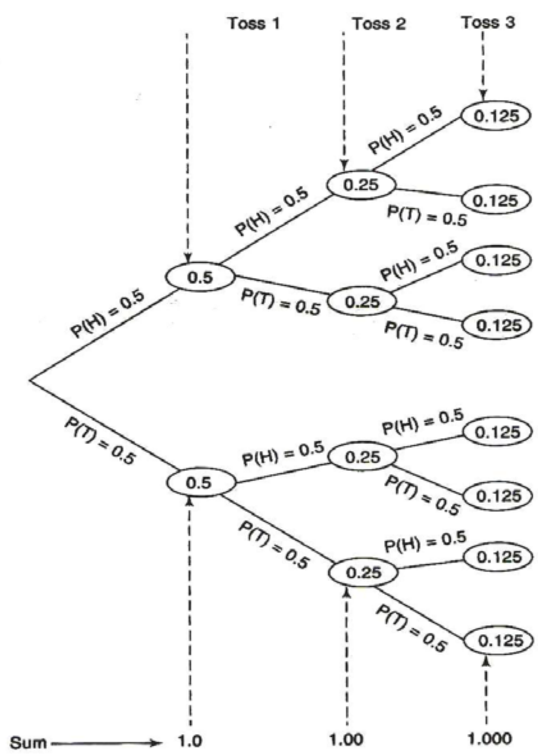


Fig 4 Probability Tree of Three Tosses

10.4 CONDITIONAL PROBABILITIES

The discussion so far was confined to two types of probabilities, marginal or unconditional probability and joint probability. Under statistical independence, only one type of probability remains to be discussed. This is known as the conditional probability.

Symbolically, conditional probability is written as P(A/B) which means that probability of event A, given that event B has occurred.

This appears to be contradictory. It may be recalled that independent events are those events whose probabilities are not affected by the occurrence of each other. This means that P(A/B) = P(A). Let us take an example to explain this.

Example: Suppose we are asked : what is the probability that the second toss of a fair coin will result in tail, given that tail resulted on the first toss?

Solution : This can be written as $P(T^2/T^1)$. It should be noted that the outcome of the first even has no influence whatsoever on the outcome of the second even since the two events are independent. The probability of a tail on the second toss is 0.5. Thus, we can write $P(T_2/T_1) = 05$.

We may now summarize the three type of probabilities under statistical independence as follows:

Types of Probability	Symbol	Formula
Marginal	P(A)	P(A)
Joint	P(AB)	P(A) * P(B)
Conditional	P(B/A)	P(B)

In order to understand conditional probability under statistical dependence, let us take an example.

Example : Suppose we have an urn containing ten balls of different colours such that

2 balls are red and dotted

1 ball is green and dotted

4 balls are red and striped

3 balls are green and striped

What is the probability of drawing any particular ball from this urn?

Solution : In all there are ten balls, each with equal probability of being drawn. The probability of drawing any particular ball from this urn is 0.1. To facilitate our discussion further, the above information is shown in the following table.

Table 1 Colour and Pattern of Ten Balls

Event	Probability of Event	
1	0.1	Red and dotted
2	0.1	
3	0.1	Green and dotted
4	0.1	Red and striped
5	0.1	
6	0.1	
7	0.1	
8	0.1	Green and striped
9	0.1	
10	0.1	

Example : Suppose we draw a ball from the urn and find it is red, what is the probability that it is striped?

Solution : Since our problem relates to red balls, we ignore the green balls completely. In all, there are six red balls of which two are dotted and four are striped. Out problem now boils down to finding the simple probabilities of dotted and striped balls. These are shown as follows ;

$$P(S/R) = 4/6 = 2/3$$

$$P(D/R) = 2/6 = \frac{1/3}{1.0}$$

It will be seen that each category of red ball has been divided by the total number of red balls. Since our problem is regarding the striped red balls, the answer is 2/3. This can be shown symbolically.

$$P(S/R) = \frac{P(SR)}{P(R)}$$

Thus, to calculate the probability of striped red balls, we divided the probability of red and striped balls by the probability of red balls. The same can be written in a generalized form as

$$P(A/B) = \frac{P(AB)}{P(B)}$$

This is the formula for calculating conditional probability under statistical dependence.

Example : What is probability of getting a dotted ball given that it is green?

Solution : We know that the total probability of green balls is 0.4 because there are four green balls out of total balls. To find the probability of the ball being dotted given that it is green, we have to divided the probability of green and dotted by the probability of green. Thus,

$$P(D/G) = \frac{P(DG)}{P(G)} = \frac{0.1}{0.4} = \frac{1}{4}$$

Similarly, we can determine the probability of drawing a green and striped ball given that it is green :

$$P(S/G) = \frac{P(SG)}{P(G)} = \frac{0.3}{0.4} = \frac{3}{4}$$

It may be noted again that the two probabilities ¼ + ¾ taken together add to

Joint Probabilities:

The formula that we used to determine conditional probability under statistical dependence is

$$P(A/B) = \frac{P(AB)}{P(B)}$$

We know that it contains one term P(AB) which, in fact, denotes joint probability. We may now rewrite this formula to determine joint probability. This can be easily done by cross multiplication.

Thus,

$$P(AB) = P(A/B) \times P(B)$$

This can be expressed as: the joint probability of events A and B is equal to the probability of event A, given that event B has already occurred, multiplied by the probability of event B.

Example: We now use this formula in our previous examples of green and red balls. Suppose we have to find the probability of red and striped ball.

Solution: $P(SR) = P(S/R) \times P(R) = 2/3 \times 6/10 = 0.4$

Similarly, we can calculate the joint probability of other events as well.

$$P(DR) = P(D/R) \times P(R) = 1/3 \times 6/10 = 0.2$$

This shown that the joint probability of dotted and red balls is equal to the product of the probability of dotted balls, given a red ball and probability of red balls. This comes to 0.2.

$$P(DG) = P(D/G) \times P(G) = \frac{1}{4} \times 4/10 = 0.1$$

This is the joint probability of dotted and green ball.

$$P(SG) = P(S/G) \times P(G) = \frac{3}{4} \times 4/10 = 0.3$$

This is the joint probability of striped and green ball.

Marginal Probabilities:

The marginal probability of the event green ball can be determined by adding the probabilities of the joint events in which green ball is contained.

Symbolically, $P(G) = P(GD) + P(GS) = 0.1 + 0.3 = 0.4$

In the same manner, we can determine the marginal probability of the event red ball by adding the probabilities of the joint events in which red ball is contained.

Symbolically, $P(R) = P(R/D) + P(RS) = 0.2 + 0.4 = 0.6$

So far, we have determined marginal probabilities of red balls and green balls. Likewise, we can determine the marginal probability of dotted balls and striped balls regardless of their colours. This has been attempted below ;

$$P(D) = P(RD) + P(GD) = 0.2 + 0.1 = 0.3$$

$$P(S) = P(RS) + P(GS) = 0.4 + 0.3 = 0.7$$

It should be noted that these two probabilities add to 1.0 as was also in the case of the earlier two calculations. The following table summarizes the probabilities under statistical dependence.

Types of Probability	Symbol	Formula
Marginal or unconditional	P(A)	P(A)
Joint	P(AB)	P(A/B) X P(B)
Conditional	P(A/B)	P(AB)/P(B)

10.5 BAYES' THEOREM

Bayes' theorem is an important statistical method, which is used in evaluating new information as well as in revising prior estimates of the probability in the light of that information. Bayes' theorem may be viewed as a means of transforming our prior probability of an event into a posterior probability of that event. Bayes' theorem, if properly used, makes it unnecessary to collect huge data over a long period in order to make good decisions on the basis of probabilities.

Example : Suppose we have two machines, I and II, which are used in the manufacture of shoes. Let E_1 be the event of shoes produced by machine I and E_2 be the event that they are produced by machine II. Machine I produces 60 percent of the shoes and machine II 40 percent. It is also reported that 10 percent of the shoes produced by machine I are defective as against the 20 percent by machine II. What is the probability that a non-defective shoe was manufactured by machine I?

Solutions : If E_1 be the event of the shoe being produced by machine I and A be the event of a non-defective shoe, our problem in symbolic terms is : $P(E_1/A)$. That is, given a non-defective shoe, what is the probability that it was produced by machine I?

From our conditional probability formulas, the probability P(E1/A) is

$$P(E_1/A) = P(E_1A) / P(A)$$

But from the theorem on total probabilities, P(A) becomes

$$P(A) = (AE_1)P + P(AE_2) = P(A/E_1)\,P(E_1) + P(A/E_2)\,P(E_2)$$

$$= \Sigma P(AE_i)P(E_i)$$

Substituting this result in (i) above, we get

$$P(E_1 / A) = \frac{P(E_1 A)}{\Sigma P(A / E_i) P(E_i)}$$

Which may also be written as

$$P(E_1 / A) = \frac{P(AE_1) P(E_1)}{\Sigma P(A / E_i) P(E_i)}$$

This is called Bayes' theorem.

It may be noted that P(E1) is the probability of a shoe being manufactured by machine I, whereas P(E1/A) is the probability of a shoe being produced by machine I, given that it is a non-defective shoe. The probability P(E1) is called prior probability and P(E1/A) is called posterior probability.

Let us set up a table to calculate the probability that a non-defective shoe was produced by machine.

Computation of Posterior Probabilities :

Event	Prior $P(E_i)$	Conditional P(A/Ei)	Joint $P(E_iA)$	Posterior $P(E_i/A)$
(1)	**(2)**	**(3)**	**(4)**	**(5) = (4)/P(A)**
Machine I (E_1)	0.6	0.9	0.54	0.54/0.86 = 0.63
Machine II (E_2)	0.4	0.8	0.32	0.32/0.86 = 0.37
Total	**1.0**		**P(A) = 0.86**	**1.00**

On the basis of the above table we can say that given a non-defective shoe, the probability that it was produced by machine I is 0.63 and the probability that it was produced by machine II is 0.37. We can see that there is some revision in the prior probabilities when we apply Bayes' theorem.

A Problem with more than Two Elementary Events: The foregoing problem related to two elementary events. Let us take a problem having three elementary events.

Example : A manufacturing firm is engaged in the production of steel pipes in its three plants with a daily production of 1,000, 1,500 and 2,500 units respectively. According to the past experience, it is known that the fractions of defective pipes produced by the three plants are respectively 0.04, 0.09 and 0.07. If a pipe is selected from a day's total production and found to be defective, find out (a) from which plant the defective pipe has come, and (b) what is the probability that it has come from the second plant?

Solution : Let the probabilities of the possible events be

$P(E_1)$ = 1,000/(1,000 + 1,500 + 2,500) = 0.2 – probability that a pipe is manufactured in plant A.

$P(E_2)$ = 1,500/(1,000 + 1,500 + 2,500) = 0.3 – probability that a pipe is manufactured in plant B.

$P(E_1)$ = 2,500/(1,000 + 1,500 + 2,500) = 0.5 – probability that a pipe is manufactured in plant C.

Let P(D) be the probability that a defective pipe is drawn. Given that the proportions of the defective pipes coming from the three plants are 0.04, 0.09 and 0.07 respectively, these are, in fact, the conditional probabilities : P(D/E1) = 0.04; $P(D/E_2)$ = 0.09; and $P(D/E_3)$ = 0.07.

Now we can multiply prior probabilities and conditional probabilities in order to obtain the joint probabilities.

Joint probabilities are

Plant A 0.04 x 0.2 = 0.008

Plant B 0.09 x 0.3 = 0.027

Plant C 0.07 x 0.5 – 0.035

Now we can obtain posterior probabilities by the following calculations :

Plant A $$\frac{0.008}{0.008+0.027+0.035} = 0.114$$

Plant B $$\frac{0.027}{0.008+0.027+0.035} = 0.386$$

Plant C $$\frac{0.035}{0.008+0.027+0.035} = 0.500$$

Table 3 : Computation of Posterior Probabilities

Event	**Prior $P(E_i)$**	**Conditional $P(E_1Ei)$**	**Joint $P(E_iA)$**	**Posterior $P(E_i/E)$**
(1)	**(2)**	**(3)**	**(4)**	**(5) = (4)/P(E)**
E1	0.2	0.04	0.04 x 0.2 = 0.008	0.008 / 0.07 = 0.11
E2	0.3	0.09	0.09 x 0.3 = 0.027	0.027/0.07 = 0.39
E3	0.5	0.07	0.07 x 0.5 = 0.035	0.035/0.07 = 050
Total	**1.0**		**P(E) = 7**	**1.00**

On the basis of these calculations, we can say that (a) most probably the defective pipe has come from plant C, and (b) the probability that the defective pipe has come from the second plant is 0.39.

10.6 PROBLEMS SOLVED

Additional Examples ;

Example : Assume that a card is randomly selected from a deck of 52 playing cards. Find the probability in each of the following cases :

a. Card drawn is the king
b. Either a heart or the queen of spades
c. Card drawn is a “diamond”.

Solution :

a. In a playing card, there are four kings. Hence, the probability of getting a king is 4/52 or 1/13.
b. There are 13 cards of “heart” and the queen of spades is 1. Hence, the required probability is (13 + 1)/52 or 7/26.
c. Here, the probability of drawing a card with “diamond” is 13/52 or ¼.

Example : Determine the probability P for each of the following events :

a. At least one head appears in two tosses of a fair coin.
b. The sum 8 appears in a single toss of a pair of fair dice.
c. An ace, a kind, a queen or ten or ‘hearts’ appears in drawing a single card from a deck of 52 playing cards.

Solution :

a. In two tosses of a fair coin, there can be four possibilities : HH, HT, TH and TT. It will be seen that H comes 3 times out of 4. Hence, P = ¾.
b. When a pair of fair dice is tossed, we can get 8 as follows :

   (2, 6), (3, 5), (4, 4), (5, 3) and (6, 2)

   As each of the six faces of one die can be associated with each of the six faces of the second die, resulting 6 x 6 = 36 cases. Hence, P = 5/36.

c. An ace, a king, a queen and ten of “hearts” add up to 4 + 4 + 4 + 1 = 13. Hence, the probability is 13/52 = ¼.

Example : A letter is chosen at random from word ‘PROFESSOR’

a. What is probability that it is a vowel?
b. What is the probability that it is a ‘S’?

Solution :

a. The word PROFESSOR contains in all 9 letters of which 3 are vowels : O, E and O. Hence, the probability that a letter is a vowel is 3/9 = $\frac{1}{3}$

Example : Suppose a fair die has its even numbered faces painted red, and the odd number faces painted white. Consider the experiment of rolling the die and the events.

A = (2 or 3 shows up)

B = (A red face shows up)

Find the following probabilities

(a) P(A) (b) P(B) (c) P(AB) (d) P(A/B) (e) P(A or B)

Solution : Since a fair die has six number 1 to 6.

As such each number has $\frac{1}{6}$ probability of occurrence.

a. Hence P(A), being 2 or 3 showing up is $\frac{1}{6}+\frac{1}{6}=\frac{2}{6}$, i.e., $\frac{1}{3}$
b. The total number of faces painted red is 3 as the die has 6 numbers.
   Hence, P(B), i.e., where a red face shows up is $\frac{3}{6}=\frac{1}{2}$
c. P (AB) is the joint probability of A and B
   Hence, P(AB) = P(A) x P(B)
   $= \frac{1}{3}\times\frac{1}{2}=\frac{1}{6}$
d. $$P(A/B) = \frac{P(AB)}{P(B)} = \frac{\frac{1}{6}}{\frac{1}{2}} \text{ or } \frac{1}{6}\times\frac{2}{1} \text{ or } \frac{1}{3}$$
e. P(A or B) = P(A) + P(B) – P(A and B)
   $= \frac{1}{3}+\frac{1}{2}-\frac{1}{6}$
   $= \frac{2+3-1}{6}=\frac{4}{6}=\frac{2}{3}$

Example : A, B and C bidding for a contract. It is believed that A has exactly half a chance that B has; B, in turn, has $4/5^{th}$ as likely as C has to gain the contract. What is the probability for each to win the contract ?

Solution : Assuming that the probability of C to gain the contract is *x*.

Then,

Probability of B to win is 4/5 of $x = 4x/5$

Probability of A to win is ½ of $4x/5 = 4x/10$

Now $4x/10 + 4x/5 + x = 1$ (Since the total of three probabilities should be 1).

Or $(20x + 40x + 50x)/50 = 1$

Or $110x = 50$

∴ $x = 50/110$ or $5/11$

Hence, the probabilities to win for C, B and A are

$C(x) = 5/11$

$B\ (4x/5) = 4 \text{ x } 5/11)/5$

$= 20/11)/5 = 4/11$

$A = (4x/10)(4\text{x}5/11)/10$

$= 20/11)/10 = 2/11$

These probabilities can also be written as C = 0.454 B = 0.364 A = 0.182

Example: Three salesmen, A, B and C have been given a target of selling 10,000 units of a particular product, the probabilities of their achieving their targets being respectively 0.25, 0.30 and 0.50. If these three salesmen try to sell the product, find the probability of success of only one salesman and failure of the other two.

Solution: Probabilities are

A	B	C
0.25	0.30	0.50

When A succeeds and B and C do not succeed, then

$P = 0.25 \text{ x } (1 - 0.30)(1 - 0.50)$

$= 0.25 \text{ x } 0.70 \text{ x } 0.50$

$= 0.0875$

Hence, the required probability that one of them succeeds and the other two do not succeed is

P = 0.0875 + 0.1125 + 0.2625

= 0.4625

Example: A sub-committee of 6 members is to be formed out of a group consisting of 7 men and 4 women. Calculate the probability that the sub-committee will consist of ss

(i) Exactly 2 women; and (ii) at least 2 women.

Solution:

(i) Out of 11 persons (7 men and 4 women) a sub-committee of 6 persons can be formed in

$$^{11}C_6 = \frac{11!}{(11-6)!6!} = \frac{11\times10\times9\times8\times7}{5\times4\times3\times2\times1} = 462 \text{ ways}$$

This is the exhaustive number of ways a sub-committee can be formed.

Number of ways for the sub-committee to consist of 4 men and 2 wome is

$$^{7}C_4 \times {}^{4}C_3 = \frac{7!}{(7-4)!4!} \times \frac{4!}{(4-3)!3!} = \frac{7\times6\times5}{3\times2} \times \frac{4\times3}{3\times2} = 140$$

Hence, probability is 140/462 or 10/33

For 2 men and 4 women

$$^{7}C_2 \times {}^{4}C_4 = \frac{7!}{(7-2)!2!} \times \frac{4!}{4!} = \frac{7\times6}{2\times1} \times \frac{4!}{4!} = 21$$

Hence, probability is 21/462 = 1/22

Probability of having at least 2 women is

$$5/11 + 10/33 + 1/22 = \frac{30+20+3}{66} = 53/66$$

Example : Two computers A and B are to be marketed. A salesman who is assigned a job of finding customers for them has 60 percent and 40 percent chances respectively of succeeding in case of computers A and B. The computers can be sold independently. Given that he was able to sell at least one computer, what is the probability that the computer A has been sold?

Solution

Let A be the event that the salesman is able to sell computer A.

Let B be the event that the salesman is able to sell computer B.

Given P(A) = 0.60 and P(B) – 4.0 and that the two events A and B are independent.

$$P(AB) = P(A).\ P(B)$$

$$= 0.60 \times 0.40 = 0.24$$

Now, probability of selling at least one computer is given by

$$P(A \text{ or } B) = P(A) + P(B) - P(AB)$$

$$= 0.60 + 0.40 - 0.24 = 0.76$$

We have to find out P(A) given P(A or B)

$$P[A/P(A \text{ or } B) = \frac{P(A)}{P(AorB)}$$

$$= \frac{0.60}{0.76}$$

$$= 0.7895$$

Example : A manufacturing firm receives shipments of machine parts from two suppliers A and B. Currently, 65 percent of parts are purchased from supplier A and the remaining from supplier B. The past record shows that 2 percent of the parts supplied by A are found defective, whereas 5 percent of the parts supplied by B are found defective. On a particular day the machine breaks down because a defective part is fitted to it.

10.7 SUMMARY

This unit focuss on conditional probability and joint probability. This unit also discusses about Bayer's Theorm.

Given the information that the part was bad, using Bayes' theorem find the probability that it was supplied by supplier B.

Solution : We have to use Baye's theorem to work out the required probability. The necessary calculations are shown in the following table.

Calculation of Probability

pplier	**Prior Probability**	**Conditional Probability**	**Joint Probability**	**Posterior (Revised) Probability**
(1)	**(2)**	**(3)**	**(4) = (2) x (3)**	**(5) = (4)/0.035**
A	0.65	0.02	0.0130	$\frac{0.0130}{0.0305} = 0.43$
B	0.35	0.05	0.0175	$\frac{0.0175}{0.0305} = 0.43$
Total			**0.0305**	**1.00**

On this basis, we come to the conclusion that the probability of getting a part of shipment that was bad and from supplier B was 0.57.

10.8 KEY WORDS

Marginal Probability

Joint Probability

Conditional Probability

10.9 SELF ASSESSMENT QUESTION

1. A jar contains black and white marbles. Two marbles are chosen without replacement. The probability of selecting a black marble and then a white marble is 0.34, and the probability of selecting a black marble on the first draw is 0.47. What is the probability of selecting a white marble on the second draw, given that the first marble drawn was black?

2. The probability that it is Friday and that a student is absent is 0.03. Since there are 5 school days in a week, the probability that it is Friday is 0.2. What is the probability that a student is absent given that today is Friday?

3. A bag contains red and blue marbles. Two marbles are drawn without replacement. The probability of selecting a red marble and then a blue marble is 0.28. The probability of selecting a red marble

draw is 0.5. What is the probability of selecting a blue marble on the second draw, given that the first marble drawn was red?

4. A committee consists of four women and three men. The committee will randomly select two people to attend a conference in Hawaii. Find the probability that both are women.

5. What is the probability that the total of two dice will be greater than 9, given that the first die is a 5?

Problems on probability are solved to provide better under standing of concept.

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UNIT 11 : THEORETICAL PROBABILITY DISTRIBUTIONS

STRUCTURE

11.0 Objectives

11.1 Introduction

11.2 Random Variable

11.3 Types of Probability Distributions

11.4 Binomial Distribution

11.5 Condition necessary for Binomial Distribution

11.6 Problems in Binomial Distribution

11.7 Poisson Distribution

11.8 Problems in Poisson Distribution

11.9 Summary

11.10 Key Words

11.11 Self Assessment Questions

11.12 References

11.0 OBJECTIVES

After studying this unit you should be able to:

* Define Random variable;
* Identify types of probability distribution;
* Explain Binomial distribution and
* Solve problems on poisson distribution.

11.1 INTRODUCTION

Here we will be looking at *probability distributions* which portray not the frequency with which values of a distribution actually occur but the probability with which we predict they will occur. Probability distributions are very important tools for modeling or representing processes that occur at random, such as customers visiting a website or accidents on a building site. A probability distribution is a table or an equation that links each outcome of a statistical experiment with its probability of occurrence.

In studying probability distributions we will look at how they can be derived and how we can *model* or represent the chances of different combinations of outcomes using the same sort of approach as we use to arrange data into frequency distributions.

A *probability distribution* is very similar to a frequency distribution. Like a frequency distribution, a probability distribution has a series of categories, but instead of categories of values it has categories of types of outcomes. The other difference is that each category has a probability instead of a frequency.

In the same way as a frequency distribution tells us how frequently each type of value occurs, a probability distribution tells us how probable each type of outcome is.

Probability Distribution Prerequisites

To understand probability distributions, it is important to understand variables. Random variables, and some notation.

A variable is a symbol (A, B, x, y, etc.) that can take on any of a specified set of values.

When the value of a variable is the outcome of a statistical experiment, that variable is a random variable.

Generally, statisticians use a capital letter to represent a random variable and a lower-case letter, to represent one of its values. For example,

X represents the random variable X.

P(X = x) refers to the probability that the random variable X is equal to a particular value, denoted by x. As an example, P(X = 1) refers to the probability that the random variable X is equal to 1.

11.2 RANDOM VARIABLE

A random variable is one that can take a number of different values, but it is not possible to know which value it does take until some experiment is performed. Usually, the experiment takes the form of drawing a sample from a population. Flipping a coin, rolling a die, and taking a sample of 600 households are some of the examples. In other words, a random variable is a description of the numeric values that the outcomes from an experiment can take. Like the 'ordinary' variables, random can be classified into two categories discrete and continuous.

A random variable is one that it said to be a discrete random variable if its possible values proceed in steps with either a finite of infinite number of steps. It can take a countable or finite number of possible values. In contrast, if a random variable represents a measurement on a continuous scale so that all values in an interval are possible, it is called a continuous random variable. Examples of a continuous random variable are price of a car and daily consumption of milk.

11.3 TYPES OF PROBABILITY DISTRIBUTIONS

There are two types of probability distributions:

• Discrete probability distributions

The probability distribution of a discrete random variable is a list of probabilities associated with each of its possible values. It is also sometimes called the probability function or the probability mass function.

More formally, the probability distribution of a discrete random variable X is a function which gives the probability f(x) that the random variable equals x, for each value x:

$f(x) = P(X=x)$

It satisfies the following conditions:

a. 0 ? f(x) ? 1

b. $\Sigma f(x) = 1$

Continuous probability distributions

Describe an "unbroken" continuum of possible occurrences. A random variable is continuous if it can take any value in an interval. The number of possible values in a range is infinite, so the Probability(of a single value) = 0

Example 1: Discrete probability distribution

The number of successful treatments out of 2 patients is discrete, because the random variable represent the number of success can be only 0, 1, or 2. The probability of all possible occurrences—P(0 successes), P(1 success), P(2 successes)—constitutes the probability distribution for this discrete random variable.

Example 2: Continuous probability distribution

The probability of a given birth weight can be anything from 3 lbs to more than 10 lbs. Thus, the random variable of birth weight is continuous, with an infinite number of possible points between any two values.

Probability Distribution of a Discrete Random Variable:

The probability distribution of a discrete random variable, say *x*, is a list of the distinct numerical values of *x* along with t heir associated probabilities. Let us take an example.

Example : Let us take x as the number of heads obtained in three tosses of a fair coin. We are required to list the numerical values of *x* along with the corresponding outcomes.

Solution : These values along with the corresponding outcomes are shown in Table 1

Table 1 : List of Outcomes

Outcome	Value of x
TTT	0
TTH	1
THT	1
THH	2
HTT	1
HTH	2
HHT	2
HHH	3

x is a variable since in three tosses of the coin, it can take any value 0, 1, 2, or 3. Further, *x* is the random variable in the sense that we could not have predicted that value of the out-come before tossing the coin. It may be noted that for each elementary outcome, there is only one value of *x*. however, as we can see, two or more elementary outcomes may give the same value.

The Expected Value of a Random Variable:

The mean of a discrete variable *x* is, in fact, the mean of its probability distribution. The mean of a discrete random variable is also called its expected value. It is denoted by E(*x*). When we perform an experiment a number of times, then what is our expectation from that experiment? The mean is the value that we expect to observe per repetition.

Example : Suppose we are given the following data relating to breakdown of a machine in a certain company during a given week, wherein *x* represents the number of breakdowns of a machine and P(x) represents the probability value of *x*.

Table 2 : Number of Probability of Breakdowns :

x	0	1	2	3	4
P(*x*)	0.12	0.20	0.25	0.30	0.13

Find out the mean number of breakdowns per week for this machine.

Solution : In order to attempt this problem, we have to multiply each value of x by its probability and then add all these products. This has been shown in table given below.

Table 3 : Calculation of Mean for the Probability Distribution of Breakdowns

x	P(*x*)	*x*.P(*x*)
0	0.12	0
1	0.20	0.20
2	0.25	0.50
3	0.30	0.90
4	0.13	0.52
	? *x* P(*x*)→	2.12

Concept of Expected Value : Thus, we find that the sum of these products gives ? [x.P(x)]=2.12, which is the mean. This can be written as μ = ? [x.P(x)] = 2.12. On the basis of this calculation, we can say that, on an average, this particular machine is expected to breakdown 2.12 times per week over a period of time. In other words, if this machine is used for several weeks, then there may not be any breakdown, for some other week there may be only one breakdown per week and so on. The mean number of breakdowns is expected to be 2.12 per week for the entire period. This is the concept of expected value.

Symbolically, E(x) = ? (x.Prob.(x)], where E(x) = Expected value of a discrete variable x and x. Prob .(x) = Product of value of variable x with its probability.

It may be noted that expected value can be derived subjectively as well. On the basis of the experimenter's own experience and judgment, one may assign probability that the random variable will take on certain values.

Let us take another example.

Example : An account of a company is hoping to receive payment from two outstanding accounts during the current month. He estimates that there is 0.6 probability of receiving Rs. 15,000 due from A and 0.75 probability of receiving Rs. 40,000 due from B. What is the expected cash flow from these two accounts?

Solution

Table 4 : Calculation of Expected Cash Flow

Account	Amount (Rs)	Probability (p_i)	Amount (x_i) (Rs)
A	15,000	0.60	9,000
B	40,000	0.75	30,000
Total expected value			39,000

Importance of Expected Value: The concept of expected value is of considerable importance to management in decision-making. This is because the criteria in decision problems involving uncertainties are usually the maximization of expected profits, or utility, and the minimization of expected costs. In Chapter 22 on Decision Theory, we shall discuss these criteria in detail giving suitable examples.

With this introduction we now turn to the binomial distribution.

11.4 BINOMIAL DISTRIBUTION

The binomial distribution is also known as the Bernoulli distribution in honour of the Swiss mathematician Jacob Bernoulli (1654-1705) who derived it.

To begin with, we go back to our frequently used examples of a fair coin in the last chapter. Assuming that the coin is tossed once, there can be two possibilities – either head (or success) or tail (or failure). The sum of the probabilities is $p + q$, where p is the probability of success and q of failure. Instead of success and failure we may also say 1 and 0.

Now, assume two coins are tossed together. Then, we can have four possibilities.

1. Both coins falling heads
2. The first coin falling head and the second falling tail
3. The first coin falling tail and the second falling head
4. Both coins falling tails.

Thus, the probabilities of 2 heads (or 2 success) = p x p = p^2.

Probabilities of one head and one tail = (p x q) = pq

Probabilities of one tail and one head = (q x p) = qp

Probabilities of 2 tails (or 2 failures) = q x q = q^2

Thus, the probabilities of 0, 1 and 2 success are given by ^{q2,} $2qp$, p^2, respectively, that is, by the successive terms of the expansion of the binomial $(q + p)^2$. In the same manner, if three coins are tossed simultaneously, probabilities of 0, 1, 2, and 3 success will respectively be given by the terms $q3$, $3q^2p$, $3qp^2$, p^3, being the successive terms of binomial $(q + p)^3$.

Let us put these results in the following form ;

For one coin or event $(q + p)^1$ that is, $q + p$

For two coin or events $(q + p)^2$ that is, $q^2 + 2qp + p^2$

For three coins or events $(q + p)^3$ that is, $q^3 + 3q^2p + 3qp^2 + p^3$

Hence, for n coins or events $(q + p)^n$

$$(q + p)^n = q^n + nq^{n-1} p + \frac{n(n-1)}{2!} q^{n-2} P^2 + \dots P^n$$

This is known as the binomial distribution

To analyze a problem using the binomial distribution you have to know the probability of each outcome and it must be the same for every trial. In other words, the results of the trials must be independent of each other.

Words like 'experiment' and 'trial' are used to describe binomial situations because of the origins and widespread use of the binomial distribution in science. Although the distribution has become widely used in many other fields, these scientific terms have stuck.

It is a discrete probability distribution. Its probability mass function is given by

$P(X) = nC_x q^{nx} p^x$, x

= 0 to n. The Binominal Distribution is given by

$(q+p)^n = q^n + nC1 \; q^{n1} \; p^1 + nC2q^{n2} \; p^2 + \dots\dots\dots\dots+p^n$

The successive terms of the expansion gives the probability of 0, 1, 2 ……..n success. The mean and variance of the distribution are np and npq. "n" and "p" are its parameters. It is a uni-modal distribution. For fixed n or p as p or n increases the distribution shifts from left to right.

Assumption under which Binomial Distribution canbe applied.

i. The experiment should be of dichotomous nature.
ii. The probability of success should remain the same from experiment to experiment.
iii. Experiments should be conducted under identical conditions.

11.5 CONDITIONS NECESSARY FOR BINOMIAL DISTRIBUTION

At this stage, we should know that there are certain conditions that must be fulfilled by a distribution if it is to be a binomial distribution. Then the conditions are ;

1. It is necessary that each observation is classified in two categories such as success and failure. For example, if raw material is obtained by a firm from its suppliers, it may be classified as defective or non-defective on the basis of its normal quality. Similarly, if a die is thrown, we may call 4, 5 or 6 success and getting 1, 2 or 3 a failure.
2. It is necessary that the probability of success (or failure) remains the same for each observation in each trial. Thus the probability of getting head (or tail) must remain the same in each toss of the experiment. In other words, if the probability of success (or failure) changes from trial or trial or if the results of each trial are classified in more than two categories, then it is not possible to use the binomial distribution.
3. The trials or individual observations must be independent of each other. In other words, no trial should influence the outcome of another trial.

11.6 PROBLEMS IN BINOMIAL DISTRIBUTION

Let us take an example. The binomial distribution $(q + p)^n$ in general terms ${}^nC_r q^{n-r} P^r$, where ${}^nC_r n!/\{r!(n-r)!\}$, where r is the number of ways in which we can get r success and n-r failures out of n trials.

Example : Find the chance of getting 3 success in 5 trials when the chance of getting a success in one trial is 2/3.

Solution : Here, $n = 5$, $p = 2/3$, $q = 1 - p = 1- 2/3 = 1/3$ and $r = 3$

Substituting these values in general terms, the required chance is

$$
\begin{aligned}
& {}^nC_r q^{n-r} P^r \\
= \quad & {}^5C_3 (1/3)^{5-3} (2/3)^3 \\
= \quad & \frac{5!}{3!(5-3)!} \times \frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \\
= \quad & \frac{5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 2 \times 1} \times \frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3}
\end{aligned}
$$

Fitting a Binomial Distribution : On the basis of some given information, if a binomial distribution is to be fitted, then the following procedure needs to be adopted.

1. Find the values of p and q. When one value is given to us, the other value can be easily obtained by subtracting the first value from 1.
2. Expand the binomial $(p+q)^n$. It may be noted that the power of n will be one less than the number of terms in the expanded binomial. For example, when n=5, there will be 6 terms.
3. Multiply each of the expanded binomial terms by the total frequency (N) so that the expected frequency in each category can be obtained. Let us take an example.

Example : Fit a binomial distribution to following data :

x	**0**	**1**	**2**	**3**	**4**
f	28	62	46	10	4

Solution :

x	f	fx
0	28	0
1	62	62
2	46	92
3	10	30
4	4	16
	150	200

$$\text{Mean} = \frac{\Sigma fx}{\Sigma f} = \frac{200}{150} = np$$

$$\therefore \qquad p = \frac{200}{150\times n} = \frac{200}{150\times 4} = \frac{200}{600} = \frac{1}{3} \qquad (\because n = 4)$$

The expected binomial frequencies can be obtained

$$f(r) = N.p(r) = N \times {}^n C_r P^r - q^{n-r}$$

$$= 150 \times {}^4 C_r \left(\frac{1}{3}\right)^r \left(\frac{2}{3}\right)^{4-r}$$

Now, to get the binomial frequencies, we have to put $r = 0, 1, 2, 3$ and 4 in the above equation. These calculations are shown in the following table.

Table.5 : Calculation of Binomial Frequencies

r	$f(r) = 150 \times {}^4C_r \left(\frac{1}{3}\right)^r \left(\frac{2}{3}\right)^{4-r}$
0	$f(0) = 150 \times {}^4C_0 \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^{4-0} = 150 \times \frac{16}{81} = 30$
1	$f(1) = 150 \times {}^4C_1 \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^{4-1} = 150 \times \frac{32}{81} = 59$
2	$f(2) = 150 \times {}^4C_2 \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^{4-2} = 150 \times \frac{24}{81} = 44$
3	$f(3) = 150 \times {}^4C_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^{4-3} = \frac{150 \times 8}{81} = 15$
4	$f(4) = 150 \times {}^4C_4 \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^{4-4} = \frac{150 \times 1}{81} = 2$

The frequencies of the binomial distribution are shown in the extreme right of the above table.

Mean and Standard Deviation of Binomial Distribution:

The mean and standard deviation of such theoretical frequency distributions where we know the number of independent events and the probability of the happening of the event in question, can be very easily calculated. If M stands for the mean of such distribution, n for the number of independent events and p for the probability of the happening of the event in a single trial, then M = np. The value of the standard deviation of the expected frequencies in such cases is

$$\sigma = \sqrt{npq}$$

Meeting the Conditions for using the Bernoulli Process

Before closing our discussion on the binomial distribution, it must be emphasized that one should be careful in using the binomial probability. It is necessary to ensure that conditions specified earlier for binomial distribution are satisfied, particularly conditions 2 and 3. Condition 2 requires that the probability of the outcome of any trial should remain unchanged for each trial. While this condition is fully met in experiments involving tossing a coin or rolling a die, in real life it may be difficult to ensure the compliance of this condition.

Condition 3 requires that the trials of a Bernoulli process must be independent of each other. This means that the outcome of one trial must not influence in any way the outcome of any other trial. This condition, too, may not be satisfied in real-life situation for example, take the case of interviewing candidates for a certain post in a company. The expert, who is interviewing the candidates, may find that t he first three candidates are far below the standard expected. In view of this, he may not remain impartial (as he was earlier) while interviewing the fourth candidate. This means violation of condition 3. One can find several situations of this type in everyday life where compliance of condition 3 becomes extremely difficult.

11.7 POISSON DISTRIBUTION

Having discussed the binomial distribution in the proceeding section, we now turn to Poisson distribution, which is also a discrete probability distribution. It was developed by a French mathematician SD Poisson (1781-1840) and hence named after him.

Along with the normal and binomial distributions, the Poisson distribution is one of the most widely used distributions. It is used in quality control statistics to count the number of defective items or in insurance problems to count the number of causalities or in waiting – time problems to count the number of incoming telephone calls or incoming customers or the number of patients arriving to consult a doctor in a given time period, and so forth. All these examples have a common feature; they can be described by a discrete random variable, which takes on integer values (0, 1, 2, 3 and so on).

The Characteristics of the Poisson distribution are:

1. The events occur independently. This means that the occurrence of a subsequent event is not at all influenced by the occurrence of an earlier event.
2. Theoretically, there is no upper limit with the number of occurrences of an event during a specified time period.
3. The probability of a single occurrence of an event within a specified time period is proportional to the length of the time period of interval.
4. In an extremely small portion of the time period, the probability of two or more occurrences of an event is negligible.

11.8 PROBLEMS IN POISSON DISTRIBUTION

Let us take an example to show how Poisson probabilities can be calculated.

Example : Suppose, we have a production process of some item that is manufactured in large quantities. We find that, n general, the proportion of defective items is p = 0.01. A random sample of 100 items is selected. What is the probability that there are 2 defective items in this sample?

Solution :

The Poisson formula is

$$P(x) = \frac{\lambda \times e^{-\lambda}}{x!}$$

Where

P(*x*)	=	Probability of x occurrences
λ^x	=	Lambda (i.e the mean number of occurrences per interval of time) raised to the *x* power
e^{λ}	=	2.71828 (being the base of the natural logarithm system), raised to the negative lambda power
x!	=	*x* factorial

Here $\lambda^x = np$ = 100 x 0.01 = 1.0

Applying the above formula to the data given

$$P(2) = \frac{(1)^2 \times (2.71828)^{-1}}{2 \times 1}$$

$$P(2) = \frac{(1)^2 \times 0.36788}{2} = 0.18394$$

Suppose, we want to know what is the probability of having upto 2 defective items in that sample of 100 items. We simply add the3 figures

P(0)	0.368
P(1)	0.368
P(2)	0.184
Total	**0.92**

The answer is 0.92

Again, if we are interested in knowing the probability of having more than 2 defective items, the answer will be

1-092 = 0.08

Example : Suppose the probability of dialing a wrong number is 0.05. Then, what is the probability of dialing exactly 3 wrong numbers in 100 dials?

Solution

$p = 0.05$

$n = 100$

$\lambda = np$

$= 100 \times 0.05 = 5$

Applying the Poisson formula,

$$P(x) = \frac{(5)^3 \times (2.71828)^{-5}}{3!}$$

$$= \frac{125 \times 0.0067*}{6} = 0.14$$

Example : Fit a Poisson distribution to the following data, which relate to the number of deaths due to the kick of a horse in 10 corps per army per annum over 20 years.

Deaths	**0**	**1**	**2**	**3**	**4**	**Total (*f*)**
Frequency	109	65	22	3	1	200

Solution : Calculate the theoretical frequencies

The theoretical expected frequencies are given by the formula

$$N \text{ x } \frac{\lambda^x \times e^{-\lambda}}{x!}$$

Where x = 0, 1, 2, 3 and 4

N = total frequency

λ = mean

e = 2.71828

In order to find the value of λ, we have to calculate the arithmetic mean.

Table : Worksheet for Data in Example 10.14

Deaths (*x*)	**Frequency (*f*)**	***fx***
0	109	0
1	65	65
2	22	44
3	3	9
4	1	4
Total	**200**	**122**

Mean = ? *fx*/*n* = 122/200 = 0.61

$$N \text{ x } \frac{\lambda^x \times e^{-\lambda}}{x!}$$

$$= \frac{200 \times (0.61)^x \times (2.71828)^{-0.61}}{x!}$$

$e^{-0.61} = 0.5435$

Now for each value of *x* from 0 to 4, we have to calculate the frequency. This is shown below :

r	***f***
0	$200 \times 0.5435 = 108.7$
1	$200 \times 0.61 \times 0.5435 = 66.3$
2	$\frac{200 \times (0.61)^2 \times 0.5435}{2} = 20.2$
3	$\frac{200 \times (0.61)^3 \times 0.5435}{3 \times 2} = 4.1$
4	$\frac{200 \times (0.61)^4 \times 0.5435}{4 \times 3 \times 2} = 0.6$

Thus, the theoretical frequencies are

Table : Theoretical Frequencies of Data in Example

x	*Tf*	*f*
0	109	109
1	66	65
2	20	22
3	4	3
4	1	1
Total	**200**	**200**

11.9 SUMMARY

This Unit deals with definition and explaination about random variable. It also describes types of probability distribution. It also gives a note on Binomial and Poission distribution. Problems on Binomial and Poission distribution are solved to give a better understanding of the concept.

11.10 KEY WORDS

Binomial Distribution

Poission Distribution

Randam Variable

11.10 SELF ASSESSMENT QUESTIONS

1. From past experience, a manager of an upscale shoe store knows that 85% of her customers will use a credit card when making purchases. Suppose three customers are in line to make a purchase.
   a) Does this example satisfy the conditions of a Bernoulli process?
   b) Construct a probability tree that delineates all possible values and their associated probabilities.
   c) Using the probability tree, derive the binomial probability distribution.
2. Approximately 20% of U.S. workers are afraid that they will never be able to retire (bankrate.com, June 23, 2008). Suppose 10 workers are randomly selected.
   a) What is the probability that none of the workers is afraid that they will never be able to retire?
   b) What is the probability that at least two of the workers are afraid that they will never be able to retire?

c) What is the probability that no more than two of the workers are afraid that they will never be able to retire?

d) Calculate the expected value, the variance, and the standard deviation of this binomial probability distribution.

3. A small life insurance company has determined that on the average it receives 6 death claims per day. Find the probability that the company receives at least seven death claims on a randomly selected day.

4. The number of traffic accidents that occurs on a particular stretch of road during a month follows a Poisson distribution with a mean of 9.4. Find the probability that less than two accidents will occur on this stretch of road during a randomly selected month.

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UNIT 12 : NORMAL PROBABILITY DISTRIBUTIONS

STRUCTURE

12.0 Objective

12.1 Introduction

12.2 Basic Definitions

12.3 Properties of Normal Distribution

12.4 The standard Normal Curve

12.5 Equation of the Standard Normal Distribution

12.6 Normal Distribution Problems

12.7 Summary

12.8 Key Words

12.9 Self Assessment Questions

12.10 References

12.0 OBJECTIVES

After studying this unit you should be able to:

* Draw Normal curve;
* Solve problems on Normal distribution and
* Appreciate Excel application of probability distribution.

12.1 INTRODUCTION

The Normal Probability Distribution is very common in the field of statistics. Whenever you measure things like people's height, weight, salary, opinions or votes, the graph of the results is very often a normal curve.

12.2 BASIC DEFINITIONS

The normal distribution refers to a family of continuous probability distributions described by the normal equation.

Normal Probability Distribution

The Normal Equation

The normal distribution is defined by the following equation:

Normal equation. The value of the random variable *Y* is:

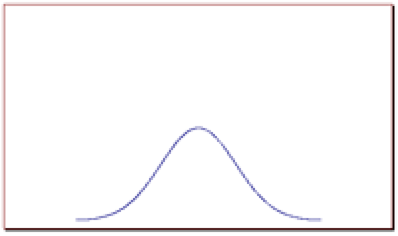
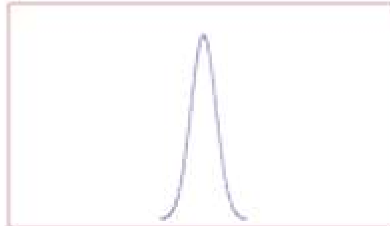
$$Y = \{ 1/[\sigma * \text{sqrt}(2\pi)] \} * e^{-(x - \mu)2/2\sigma 2}$$

where *X* is a normal random variable, μ is the mean, σ is the standard deviation, π is approximately 3.14159, and *e* is approximately 2.71828.

The random variable *X* in the normal equation is called the **normal random variable**. The normal equation is the probability density function for the normal distribution.

The Normal Curve

The graph of the normal distribution depends on two factors - the mean and the standard deviation. The mean of the distribution determines the location of the center of the graph, and the standard deviation determines the height and width of the graph. When the standard deviation is large, the curve is short and wide; when the standard deviation is small, the curve is tall and narrow. All normal distributions look like a symmetric, bell-shaped curve, as shown below.

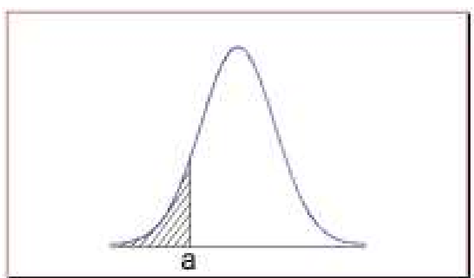
 

The curve on the left is shorter and wider than the curve on the right, because the curve on the left has a bigger standard deviation.

Probability and the Normal Curve

The normal distribution is a continuous probability distribution. This has several implications for probability.

- The total area under the normal curve is equal to 1.
- The probability that a normal random variable X equals any particular value is 0.
- The probability that X is greater than a equals the area under the normal curve bounded by a and plus infinity (as indicated by the ***non-shaded*** area in the figure below).
- The probability that X is less than a equals the area under the normal curve bounded by a and minus infinity (as indicated by the ***shaded*** area in the figure below).



Additionally, every normal curve (regardless of its mean or standard deviation) conforms to the following "rule".

- About 68% of the area under the curve falls within 1 standard deviation of the mean.
- About 95% of the area under the curve falls within 2 standard deviations of the mean.
- About 99.7% of the area under the curve falls within 3 standard deviations of the mean.

Collectively, these points are known as the **empirical rule** or the **68-95-99.7 rule**. Clearly, given a normal distribution, most outcomes will be within 3 standard deviations of the mean.

To find the probability associated with a normal random variable, use a graphing calculator, an online normal distribution calculator, or a normal distribution table. In the examples below, we illustrate the use of Stat Trek's Normal Distribution Calculator, a free tool available on this site. In the next lesson, we demonstrate the use of normal distribution tables.

Example 1

An average light bulb manufactured by the Acme Corporation lasts 300 days with a standard deviation of 50 days. Assuming that bulb life is normally distributed, what is the probability that an Acme light bulb will last at most 365 days?

Solution: Given a mean score of 300 days and a standard deviation of 50 days, we want to find the cumulative probability that bulb life is less than or equal to 365 days. Thus, we know the following:

§ The value of the normal random variable is 365 days.

§ The mean is equal to 300 days.

§ The standard deviation is equal to 50 days.

We enter these values into the Normal Distribution Calculator and compute the cumulative probability. The answer is: $P(X \leq 365) = 0.90$. Hence, there is a 90% chance that a light bulb will burn out within 365 days.

Example 2

Suppose scores on an IQ test are normally distributed. If the test has a mean of 100 and a standard deviation of 10, what is the probability that a person who takes the test will score between 90 and 110?

Solution: Here, we want to know the probability that the test score falls between 90 and 110. The "trick" to solving this problem is to realize the following:

$$P(90 < X < 110) = P(X < 110) - P(X < 90)$$

We use the Normal Distribution Calculator to compute both probabilities on the right side of the above equation.

§ To compute $P(X < 110)$, we enter the following inputs into the calculator: The value of the normal random variable is 110, the mean is 100, and the standard deviation is 10. We find that $P(X < 110)$ is 0.84.

§ To compute P(X < 90), we enter the following inputs into the calculator: The value of the normal random variable is 90, the mean is 100, and the standard deviation is 10. We find that P(X < 90) is 0.16.

We use these findings to compute our final answer as follows:

$$P(90 < X < 110) = P(X < 110) - P(X < 90)$$
$$P(90 < X < 110) = 0.84 - 0.16$$
$$P(90 < X < 110) = 0.68$$

Thus, about 68% of the test scores will fall between 90 and 110.

12.3 PROPERTIES OF NORMAL DISTRIBUTION

Figure 1 shows the normal probability distribution :

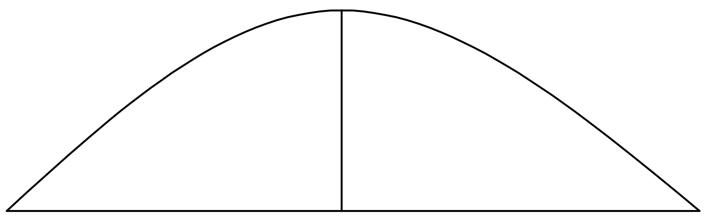


Fig 1 : The Normal Probability Distribution

Let us see what does this figure indicate in terms of characteristics of the normal distribution. It indicates the following characteristics.

1. The curve is bell-shaped, that is, it has the same shape on either side of the vertical line from mean.
2. It has a single peak. As such it is unimodal.
3. The mean is located at the centre of the distribution.
4. The distribution is symmetrical
5. The two tails of the distribution extend indefinitely but never touch the horizontal axis.
6. Since the normal curve is symmetrical, the lower and upper quartiles are equidistant from the median, that is, Q_3- Median = Median – Q_1.
7. The mean, median and mode have the same value, that is, mean = median = mode.
8. The percentage distribution of area under standard normal curve is broadly as follows : ±1ó 68.27%; ±2ó 95.44% and ± 3ó 99.73%. This was also shown in Fig 7.1.

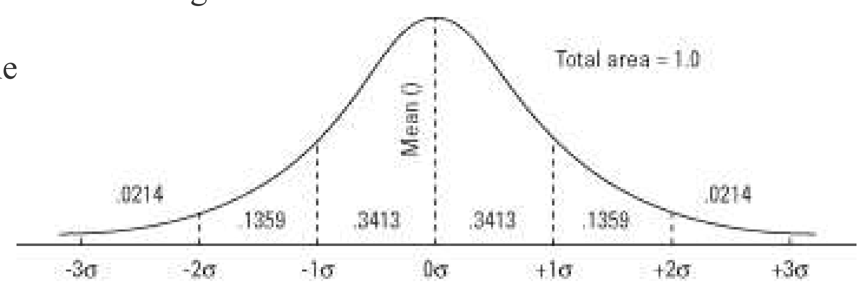
The units for the standard normal distribution curve are denoted by Z and are called the Z values or Z scores. They are also called standard units or standard scores. The Z scores is known as a 'standardized' variable because it has a zero mean and a standard deviation of one.

As can be seen from Fig. 10.3 the horizontal axis is labeled *Z*. The *Z* values on the right side of the mean are positive while those on its left side are negative. The *Z* for a point on the horizontal axis gives the distance between the mean and that point in terms of the standard deviation. For example, a specific value of *Z* gives the distance between the mean and the point represented by *Z* in terms of 1 standard deviation to the right of the mean. Likewise, a point with a value of $Z = -1$ is one standard deviation to the left of the mean. It can be seen that the mean is at the centre and its value has been shown as zero. The area on either side of the mean is 05. Thus, the total area under the curve is 1.

124 THE STANDARD NORMAL CURVE

When the area of the standard normal curve is divided into sections by standard deviations above and below the mean, the area in each section is a known quantity (see Figure 2). As explained earlier, the area in each section is the same as the probability of randomly drawing a value in that range.

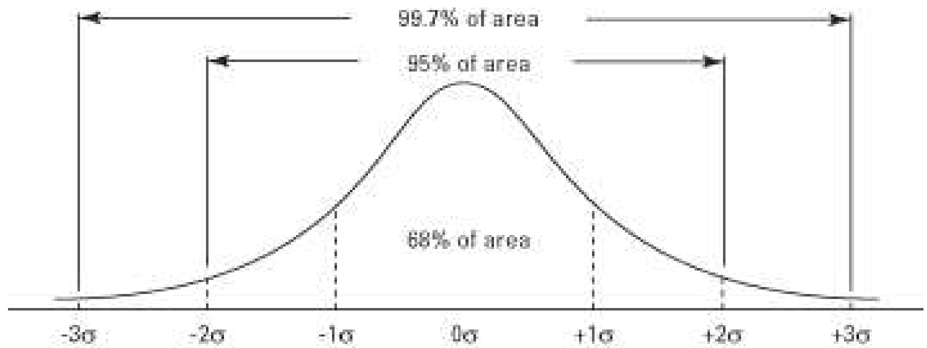
Figure 2.The



For example, 0.3413 of the curve falls between the mean and one standard deviation above the mean, which means that about 34 percent of all the values of a normally distributed variable are between the mean and one standard deviation above it. It also means that there is a 0.3413 chance that a value drawn at random from the distribution will lie between these two points.

Sections of the curve above and below the mean may be added together to find the probability of obtaining a value within (plus or minus) a given number of standard deviations of the mean (see Figure 3). For example, the amount of curve area between one standard deviation above the mean and one standard deviation below is 0.3413 + 0.3413 = 0.6826,

Figure3.The normal curve and the area under the curve between ó units.



In order to use the area of the normal curve to determine the probability of occurrence of a given value, the value must first be **standardized,** or converted to a *z*-**score** . To convert a value to a *z*-score is to express it in terms of how many standard deviations it is above or below the mean. After the *z*-score is obtained, you can look up its corresponding probability in a table. The formula to compute a*z*-score is

$$z = \frac{x - \mu}{\sigma}$$

where ***x*** is the value to be converted, μ is the population mean, and σ is the population standard deviation.

Example 1

A normal distribution of retail-store purchases has a mean of $14.31 and a standard deviation of 6.40. What percentage of purchases were under $10? First, compute the z-score:

$$z = \frac{10 - 14.31}{6.40} = -0.67$$

The next step is to look up the *z*-score in the table of standard normal probabilities (see Table 2 in "Statistics Tables"). The standard normal table lists the probabilities (curve areas) associated with given *z*-scores.

Table 2 in "Statistics Tables" gives the area of the curve below *z*—in other words, the probability of obtaining a value of *z* or lower. Not all standard normal tables use the same format, however. Some list only positive *z*-scores and give the area of the curve between the mean and *z*. Such a table is slightly more difficult to use, but the fact that the normal curve is symmetric makes it possible to use it to determine the probability associated with any

z-score, and vice versa.

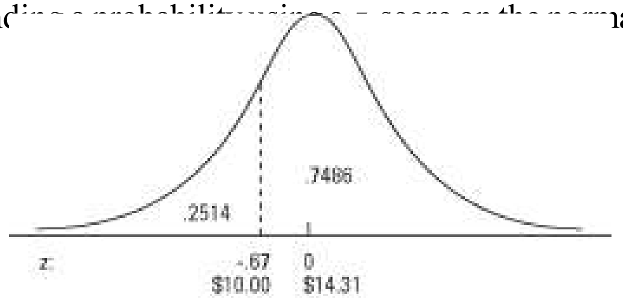
To use Table 2 (the table of standard normal probabilities) in “Statistics Tables,” first look up the z-score in the left column, which lists z to the first decimal place. Then look along the top row for the second decimal place. The intersection of the row and column is the probability. In the example, you first find –0.6 in the left column and then 0.07 in the top row. Their intersection is 0.2514. The answer, then, is that about 25 percent of the purchases were under \$10 (see Figure 3).

What if you had wanted to know the percentage of purchases above a certain amount? Because Table gives the area of the curve below a given z, to obtain the area of the curve above z, simply subtract the tabled probability from 1. The area of the curve above a z of –0.67 is 1 – 0.2514 = 0.7486. Approximately 75 percent of the purchases were above \$10.

Just as Table

can be used to obtain probabilities from z-scores, it can be used to do the reverse.

Figure 3.Fin[illegible]al curve.



Example 2

Using the previous example, what purchase amount marks the lower 10 percent of the distribution?

Locate in Table

the probability of 0.1000, or as close as you can find, and read off the corresponding z-score. The figure that you seek lies between the tabled probabilities of 0.0985 and 0.1003, but closer to 0.1003, which corresponds to a z-score of –1.28. Now, use the z formula, this time solving for x:

$$\frac{x-14.31}{6.4}=-1.28$$
$$x-14.31=(-1.28)(6.4)$$
$$x=-8.192+14.31$$
$$=6.118$$

Approximately 10 percent of the purchases were below \$6.12.

12.5 EQUATION OF THE STANDARD NORMAL DISTRIBUTION

The **standard normal distribution** is a special case of the normal distribution. It is the distribution that occurs when a normal random variable has a mean of zero and a standard deviation of one.

The normal random variable of a standard normal distribution is called a **standard score** or a **z-score**. Every normal random variable X can be transformed into a z score via the following equation:

$$z = (X - \mu) / \sigma$$

where X is a normal random variable, μ is the mean mean of X, and σ is the standard deviation of X.

Standard Normal Distribution Table

A **standard normal distribution table** shows a cumulative probability associated with a particular z-score. Table rows show the whole number and tenths place of the z-score. Table columns show the hundredths place. The cumulative probability (often from minus infinity to the z-score) appears in the cell of the table.

For example, a section of the standard normal table is reproduced below. To find the cumulative probability of a z-score equal to -1.31, cross-reference the row of the table containing -1.3 with the column containing 0.01. The table shows that the probability that a standard normal random variable will be less than -1.31 is 0.0951; that is, P(Z < -1.31) = 0.0951.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
...	...	...	...	...	...	...	...	...	...	...
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0722	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
...	...	...	...	...	...	...	...	...	...	...
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Of course, you may not be interested in the probability that a standard normal random variable falls between minus infinity and a given value. You may want to know the probability that it lies between a given value and plus infinity. Or you may want to know the probability that a standard normal random variable lies between two given values. These probabilities are easy to compute from a normal distribution table. Here's how.

- Find $P(Z > a)$. The probability that a standard normal random variable (z) is greater than a given value (a) is easy to find. The table shows the $P(Z < a)$. The $P(Z > a) = 1 - P(Z < a)$.

  Suppose, for example, that we want to know the probability that a z-score will be greater than 3.00. From the table (see above), we find that $P(Z < 3.00) = 0.9987$. Therefore, $P(Z > 3.00) = 1 - P(Z < 3.00) = 1 - 0.9987 = 0.0013$.

- Find $P(a < Z < b)$. The probability that a standard normal random variables lies between two values is also easy to find. The $P(a < Z < b) = P(Z < b) - P(Z < a)$. For example, suppose we want to know the probability that a z-score will be greater than -1.40 and less than -1.20. From the table (see above), we find that $P(Z < -1.20) = 0.1151$; and $P(Z < -1.40) = 0.0808$. Therefore, $P(-1.40 < Z < -1.20) = P(Z < -1.20) - P(Z < -1.40) = 0.1151 - 0.0808 = 0.0343$.

In school or on the Advanced Placement Statistics Exam, you may be called upon to use or interpret standard normal distribution tables. Standard normal tables are commonly found in appendices of most statistics texts.

The Normal Distribution as a Model for Measurements

Often, phenomena in the real world follow a normal (or near-normal) distribution. This allows researchers to use the normal distribution as a model for assessing probabilities associated with real-world phenomena. Typically, the analysis involves two steps.

§ Transform raw data. Usually, the raw data are not in the form of z-scores. They need to be transformed into z-scores, using the transformation equation presented earlier: *z* = (*X* - ì) / ó.

§ Find probability. Once the data have been transformed into z-scores, you can use standard normal distribution tables, online calculators (e.g., Stat Trek's free normal distribution calculator), or handheld graphing calculators to find probabilities associated with the z-scores.

The problem in the next section demonstrates the use of the normal distribution as a model for measurement.

Problem 1

Molly earned a score of 940 on a national achievement test. The mean test score was 850 with a standard deviation of 100. What proportion of students had a higher score than Molly? (Assume that test scores are normally distributed.)

(A) 0.10
(B) 0.18
(C) 0.50
(D) 0.82
(E) 0.90

Solution

The correct answer is B. As part of the solution to this problem, we assume that test scores are normally distributed. In this way, we use the normal distribution as a model for measurement. Given an assumption of normality, the solution involves three steps.

§ First, we transform Molly's test score into a z-score, using the z-score transformation equation.

$z = (X - \mu) / \sigma = (940 - 850) / 100 = 0.90$

- Then, using an online calculator (e.g., Stat Trek's free normal distribution calculator), a handheldgraphing calculator, or the standard normal distribution table, we find the cumulative probability associated with the z-score. In this case, we find P(Z < 0.90) = 0.8159.
- Therefore, the P(Z > 0.90) = 1 - P(Z < 0.90) = 1 - 0.8159 = 0.1841.

Thus, we estimate that 18.41 percent of the students tested had a higher score than Molly.

12.6 NORMAL DISTRIBUTION PROBLEMS

Problems and applications on normal distributions are presented. The answers to these problems are at the bottom of the page. X is a normally normally distributed variable with mean $\mu = 30$ and standard deviation $\sigma = 4$. Find

a) P(x < 40)
b) P(x > 21)
c) P(30 < x < 35)

ANS: Note: What is meant here by area is the area under the standard normal curve.

a) For x = 40, the z-value z = (40 - 30) / 4 = 2.5
Hence P(x < 40) = P(z < 2.5) = [area to the left of 2.5] = 0.9938

b) For x = 21, z = (21 - 30) / 4 = -2.25
Hence P(x > 21) = P(z > -2.25) = [total area] - [area to the left of -2.25]
= 1 - 0.0122 = 0.9878

c) For x = 30 , z = (30 - 30) / 4 = 0 and for x = 35, z = (35 - 30) / 4 = 1.25
Hence P(30 < x < 35) = P(0 < z < 1.25) = [area to the left of z = 1.25] - [area to the left of 0]
= 0.8944 - 0.5 = 0.3944

1. A radar unit is used to measure speed of cars on a motorway. The speed is normally distributed with a mean of 90 km/hr and a standard deviation of 10 km/hr. What is the probability that a car picked at random is travelling at more than 100 km/hr?

Ans: Let x be the random variable that represents the speed of cars. x has ì = 90 and ó = 10. We have to find the probability that x is higher than 100 or P(x > 100)

For x = 100 , z = (100 - 90) / 10 = 1

P(x > 90) = P(z >, 1) = [total area] - [area to the left of z = 1]

= 1 - 0.8413 = 0.1587

The probability that a car selected at a random has a speed greater than 100 km/hr is equal to 0.1587

2. For a certain type of computers, the length of time bewteen charges of the battery is normally distributed with a mean of 50 hours and a standard deviation of 15 hours. John owns one of these computers and wants to know the probability that the length of time will be between 50 and 70 hours.

ANS: Let x be the random variable that represents the length of time. It has a mean of 50 and a standard deviation of 15. We have to find the probability that x is between 50 and 70 or P(50< x < 70)

For x = 50 , z = (50 - 50) / 15 = 0

For x = 70 , z = (70 - 50) / 15 = 1.33 (rounded to 2 decimal places)

P(50< x < 70) = P(0< z < 1.33) = [area to the left of z = 1.33] - [area to the left of z = 0]

= 0.9082 - 0.5 = 0.4082

The probability that John's computer has a length of time between 50 and 70 hours is equal to 0.4082.

3. Entry to a certain University is determined by a national test. The scores on this test are normally distributed with a mean of 500 and a standard deviation of 100. Tom wants to be admitted to this university and he knows that he must score better than at least 70% of the students who took the test. Tom takes the test and scores 585. Will he be admitted to this university?

Ans: Let x be the random variable that represents the scores. x is normally ditsributed with a mean of 500 and a standard deviation of 100. The total area under the normal curve represents the total number of students who took the test. If we multiply the values of the areas under the curve by 100, we obtain percentages.

For x = 585 , z = (585 - 500) / 100 = 0.85

The proportion P of students who scored below 585 is given by

P = [area to the left of z = 0.85] = 0.8023 = 80.23%

Tom scored better than 80.23% of the students who took the test and he will be admitted to this University.

1. The annual salaries of employees in a large company are approximateley normally distributed with a mean of $50,000 and a standard deviation of $20,000.

    a) What percent of people earn less than $40,000?

    b) What percent of people earn between $45,000 and $65,000?

    c) What percent of people earn more than $70,000?

ans) For x = 40000, z = -0.5

Area to the left (less than) of z = -0.5 is equal to 0.3085 = 30.85% earn less than $40,000.

b) For x = 45000 , z = -0.25 and for x = 65000, z = 0.75

Area between z = -0.25 and z = 0.75 is equal to 0.3720 = 37.20 earn between $45,000 and $65,000.

c)For x = 70000, z = 1

Area to the right (higher) of z = 1 is equal to 0.1586 = 15.86% earn more than $70,000.

12.7 SUMMARY

This unit focuses on normal distribution. The details of normal curve are given here. Problems on normal distribution are also solved in this unit.

12.8 KEY WORDS

Normal Curve

Normal Distribution

Normal Distribution table

12.8 SELF ASSESSMENT QUESTIONS

1. An average light bulb manufactured by the Acme Corporation lasts 300 days with a standard deviation of 50 days. Assuming that bulb life is normally distributed, what is the probability that an Acme light bulb will last at most 365 days?
2. Suppose scores on an IQ test are normally distributed. If the test has a mean of 100 and a standard deviation of 10, what is the probability that a person who takes the test will score between 90 and 110?

12.9 REFERENCES

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